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SOME RESULTS ON POHLKE'S TYPE ELLIPSES

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Abstract. We give here formulae for determining the Pohlke's ellipse and the secondary Pohlke's ellipse of a triad of segments in a plane. Then we apply these results to find an explicit expression of the secondary Pohlke's projection introduced in [6].

1. Introduction

Let OP_1, OP_2, OP_3 be three non-parallel segments in a plane ω and let $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$ be the concentric ellipses defined by the three pairs of conjugate semi-diameters (OP_1, OP_2), (OP_2, OP_3) and (OP_3, OP_1) respectively. It was proved in [8] and then in [6] that there are at most two distinct ellipses with center O circumscribing $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$.
The first, which we denote by $\mathcal{E}_{\mathtt{P}}$, is the Pohlke's ellipse (see also [2], [3]). It is determined by the requirement that there exists a sphere S with center O, three points $Q_1, Q_2, Q_3 \in S$ and a parallel projection $\Pi : \mathbb{E}^3 \to \omega$ (i.e., a Pohlke's projection) such that:

$$\Pi(OQ_i) = OP_i \quad (1 \le i \le 3), \tag{1}$$

$$OQ_1 \perp OQ_2, \quad OQ_2 \perp OQ_3, \quad OQ_3 \perp OQ_1. \tag{2}$$

With S, Π as above, the Pohlke's ellipse $\mathcal{E}_{\mathtt{P}}$ for OP_1, OP_2, OP_3 is the contour of the projection onto ω of the sphere S, i.e.

$$\mathcal{E}_{\mathtt{P}} \overset{\text{def}}{=} \Pi(S \cap \pi), \tag{3}$$

where π the plane through O and perpendicular to the direction of Π. Existence and uniqueness of such an ellipse are guaranteed by Pohlke's theorem of oblique axonometry [7]. See [1], [4] for an analytic proof. The other, which we denote by $\mathcal{E}_{\mathtt{S}}$, is the *secondary* Pohlke's ellipse:

Definition 1.1. *A secondary Pohlke's ellipse for the triad of segments OP_1, OP_2, OP_3 is an ellipse $\mathcal{E}_{\mathtt{S}} \neq \mathcal{E}_{\mathtt{P}}$, centered at O, which circumscribes the three ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$.*

By the results of [6] (Theorem 2.1, (**a**) ⇔ (**b**)) a secondary Pohlke's ellipse $\mathcal{E}_{\mathtt{S}}$ is determined by the requirement that there exists a sphere $\widetilde{S}$ with center O, three points $R_1, R_2, R_3 \in \widetilde{S}$ and a parallel projection $\widetilde{\Pi} : \mathbb{E}^3 \to \omega$ (i.e., a secondary Pohlke's projection) such that:

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$$\widetilde{\Pi}(OR_i) = OP_i \quad (1 \le i \le 3), \tag{4}$$

$$OR_1 \perp OR_2\,, \quad OR_2 \perp OR_3 \quad \text{and} \quad OR_3 \perp OR_1', \tag{5}$$

$$R_i \notin \widetilde{\pi} \quad (\text{i.e., } R_i \ne R_i') \quad (1 \le i \le 3) \tag{6}$$

where $\widetilde{\pi}$ is the plane through O and perpendicular to the direction of $\widetilde{\Pi}$; the point R_i' is symmetric to R_i with respect to $\widetilde{\pi}$. With $\widetilde{S}$, $\widetilde{\Pi}$ and $\widetilde{\pi}$ as above, we have

$$\mathcal{E}_{\mathtt{S}} = \widetilde{\Pi}(\widetilde{S} \cap \widetilde{\pi}). \tag{7}$$

See also [8] for an alternative approach.

Unlike the Pohlke's ellipse $\mathcal{E}_{\mathtt{P}}$, which exists even if two of the segments OP_1, OP_2, OP_3 are parallel (see Section 2), the secondary Pohlke's ellipse $\mathcal{E}_{\mathtt{S}}$ does not always exist. More precisely, from [6] (Theorem 2.1, equivalence (**a**), (**b**) $\Leftrightarrow$ (**c**)) we also know that:

Theorem 1.1. *Suppose the segments OP_1, OP_2, OP_3 are non-parallel. Then there exists a secondary Pohlke's ellipse $\mathcal{E}_{\mathtt{S}}$ if and only if*

$$a\,\overrightarrow{OP_1} + b\,\overrightarrow{OP_2} + c\,\overrightarrow{OP_3} = 0, \tag{8}$$

with $a, b, c \ne 0$ such that

$$G(a,b,c) \overset{def}{=} a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 > 0. \tag{9}$$

Further, if $\mathcal{E}_{\mathtt{S}}$ exists then $\mathcal{E}_{\mathtt{S}}$ is unique.

The preceding definitions of $\mathcal{E}_{\mathtt{P}}$ and $\mathcal{E}_{\mathtt{S}}$ are not invariant under affine transformations of the euclidean space $\mathbb{E}^3$ due to the requirement that S, $\widetilde{S}$ be spheres and also for the orthogonality conditions in (2) and (5), (6). However, we show here that under affine transformation of the plane ω the Pohlke's ellipse of the segments OP_1, OP_2, OP_3 transforms into the Pohlke's ellipse of the transformed segments and the same is true for the secondary Pohlke's ellipse when it exists, i.e., if (8)-(9) holds.

Notation 1.1. *For greater clarity we will often write*

$$\mathcal{E}_{\mathtt{P}}(O, P_1, P_2, P_3) \quad \textit{and} \quad \mathcal{E}_{\mathtt{S}}(O, P_1, P_2, P_3), \tag{10}$$

instead of $\mathcal{E}_{\mathtt{P}}$ and $\mathcal{E}_{\mathtt{S}}$, to make explicit the triad of segments from which a given Pohlke's ellipse or a given secondary Pohlke's ellipse refers.

In this article we will demonstrate a number of facts about Pohlke's ellipses and secondary Pohlke's ellipses which we can summarize as follows:

- In Section 3, assuming OP_1, OP_2, OP_3 are not all parallel, we determine a pair of conjugate semi-diameters of the Pohlke's ellipse $\mathcal{E}_{\mathtt{P}}$ and then we apply this result to prove that if $\Psi : \omega \to \omega$ is any affine transformation then

$$\Psi(\mathcal{E}_{\mathtt{P}}(O, P_1, P_2, P_3)) = \mathcal{E}_{\mathtt{P}}(\Psi(O), \Psi(P_1), \Psi(P_2), \Psi(P_3)). \tag{11}$$

- In Section 4, assuming OP_1, OP_2, OP_3 are non-parallel and (8)-(9) holds, we demonstrate similar results for the secondary Pohlke's ellipse $\mathcal{E}_{\mathtt{S}}$. In particular, noting that $\mathcal{E}_{\mathtt{S}}(\Psi(O), \Psi(P_1), \Psi(P_2), \Psi(P_3))$

exists because condition (8)-(9) is invariant under affine transformations of the plane ω, we prove that

$$\Psi(\mathcal{E}_{\mathtt{S}}(O,P_1,P_2,P_3)) = \mathcal{E}_{\mathtt{S}}(\Psi(O),\Psi(P_1),\Psi(P_2),\Psi(P_3)). \tag{12}$$

Using (11) and (12) we also show that

$$\text{area}(\mathcal{E}_{\mathtt{P}}) < \text{area}(\mathcal{E}_{\mathtt{S}}), \tag{13}$$

because it holds if two of the segments OP_1, OP_2, OP_3 are perpendicular and equal.

- In Section 5, assuming OP_1, OP_2, OP_3 are non-parallel and (8)-(9) holds, we show that

$$\mathcal{E}_{\mathtt{S}}(O,P_1,P_2,P_3) = \mathcal{E}_{\mathtt{P}}(O,P_1,P_2,X_3), \tag{14}$$

where the point X_3 is such that

$$\pm\overrightarrow{OX_3} = \frac{a(a^2-b^2-c^2)}{c\sqrt{G}}\,\overrightarrow{OP_1} + \frac{b(a^2-b^2+c^2)}{c\sqrt{G}}\,\overrightarrow{OP_2}, \tag{15}$$

with $G = G(a,b,c)$ the quantity defined by (9).
Similarly we can prove that $\mathcal{E}_{\mathtt{S}}(O,P_1,P_2,P_3) = \mathcal{E}_{\mathtt{P}}(O,X_1,P_2,P_3) = \mathcal{E}_{\mathtt{P}}(O,P_1,X_2,P_3)$ by appropriately defining X_1, X_2 respectively.

- In Section 5.1, applying (14) and the formulae of [4] for Pohlke's projection, we finally give a procedure to explicitly determine the secondary Pohlke's projection $\widetilde{\Pi}$ and the points R_1, R_2, R_3 such that conditions (4), (5), (6) hold.

2. Preliminaries

In this section we suppose OP_1, OP_2, OP_3 are not all parallel. [1] To determine the Pohlke's ellipse $\mathcal{E}_{\mathtt{P}}$ we resume some of the arguments introduced in [4], [5]. Namely, we adopt a system of coordinate axes x, y, z such that ω is the plane $z = 0$,

$$O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} x_3 \\ y_3 \\ 0 \end{pmatrix} \tag{16}$$

and we also consider the matrix

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ 0 \end{pmatrix}. \tag{17}$$

The rows A_1, A_2 are linearly independent (i.e., $\text{car}(A) = 2$) because OP_1, OP_2, OP_3 are not all parallel. Hence can we define:

$$\gamma \overset{\text{def}}{=} \arccos\left(\frac{A_1 \cdot A_2}{\|A_1\|\,\|A_2\|}\right), \quad \lambda \overset{\text{def}}{=} \frac{\|A_1\|}{\|A_2\|}. \tag{18}$$

[1] If two of the segments OP_1, OP_2, OP_3 are parallel (or if one of them vanishes) we can still say that $\mathcal{E}_{\mathtt{P}}$ circumscribes $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$ but we need to introduce *degenerate* ellipses as in [1, pp. 372-373]. For instance, if $OP_1 \parallel OP_2$ then we set $\mathcal{E}_{P_1,P_2} = MN$, where MN is the segment parallel to OP_1, OP_2 such that $O = (M+N)/2$ and $|ON|^2 = |OP_1|^2 + |OP_2|^2$. In this case we say that $\mathcal{E}_{\mathtt{P}}$ circumscribes $\mathcal{E}_{P_1,P_2}$ if $M, N \in \mathcal{E}_{\mathtt{P}}$. We also say that $\mathcal{E}_{\mathtt{P}}$ is tangent to $\mathcal{E}_{P_1,P_2}$ at M, N. See the Definitions 3.1, 3.3 of [6].

Noting that $0 < \gamma < \pi$ and $\lambda > 0$, we can also introduce the quantities:

$$\eta \overset{\text{def}}{=} \frac{\lambda^2 + 1 + \sqrt{(\lambda^2+1)^2 - 4\lambda^2 \sin^2\gamma}}{2\lambda^2 \sin^2\gamma} \tag{19}$$

and then [2]

$$(\alpha, \beta) \overset{\text{def}}{=} \pm \Big(\sqrt{\eta\, \lambda^2 - 1}\ ,\ \operatorname{sgn}(\cos\gamma)\sqrt{\eta - 1}\ \Big), \tag{20}$$

where

$$\operatorname{sgn}(t) \overset{\text{def}}{=} \begin{cases} 1 & \text{if} \quad t \geq 0 \\ -1 & \text{if} \quad t < 0 \end{cases}. \tag{21}$$

Finally, we define the parallel projection $\Pi : \mathbb{E}^3 \to \omega$ as

$$\Pi \begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} x + \alpha z \\ y + \beta z \\ 0 \end{pmatrix}. \tag{22}$$

The Pohlke's ellipse $\mathcal{E}_{\mathrm{P}}$ of OP_1, OP_2, OP_3 is then the contour of the projection into the plane ω of the sphere S with center O and radius

$$\rho \overset{\text{def}}{=} \frac{\|A_1\|}{\lambda\sqrt{\eta}} = \frac{\|A_2\|}{\sqrt{\eta}}\,. \tag{23}$$

Namely, $\mathcal{E}_{\mathrm{P}} = \Pi(S \cap \pi)$ where π is the plane $\pi : \alpha x + \beta y - z = 0$.

Remark 2.1. *It is worthwhile noting that $\mathcal{E}_{\mathrm{P}}$ uniquely determines the sphere S centered at O, because the radius of S must be equal to the semi-minor axis of $\mathcal{E}_{\mathrm{P}}$. Furthermore, the Pohlke's projection $\Pi : \mathbb{E}^3 \to \omega$ is determined up to symmetry with respect to the plane ω. Namely, if the semi-axes of $\mathcal{E}_{\mathrm{P}}$ are given by two perpendicular segments $OV, OW \subset \omega$ such that*

$$0 < |OV| \leq |OW| \quad \textit{and} \quad W = \begin{pmatrix} p \\ q \\ 0 \end{pmatrix}, \tag{24}$$

then the sphere S has radius $\rho = |OV|$ and the direction of projection is given by the column vector

$$\overrightarrow{n} = \begin{pmatrix} \delta p \\ \delta q \\ \pm 1 \end{pmatrix} \quad \textit{with} \quad \delta = \sqrt{\frac{p^2 + q^2 - \rho^2}{\rho^2 (p^2 + q^2)}}\,. \tag{25}$$

If $\delta = 0$ then $\mathcal{E}_{\mathrm{P}}$ is a circle and we have only the orthogonal projection. Conversely, if $\delta > 0$ the two possible signs of the last component of $\overrightarrow{n}$ correspond to two distinct projections which are symmetric with respect to the plane ω. Indeed, if $\overline{\Pi} : \mathbb{E}^3 \to \omega$ is defined by

$$\overline{\Pi}(P) = \Pi(\overline{P}) \quad \textit{where } \overline{P} \textit{ is symmetric to } P \textit{ with respect to } \omega, \tag{26}$$

[2] We note that $\eta,\ \eta\lambda^2 \geq 1$. Indeed from (19) we easily have:

$$\eta(\lambda, \gamma) \geq \eta(\gamma, \frac{\pi}{2}) = \frac{\lambda^2 + 1 + |\lambda^2 - 1|}{2\lambda^2} = \begin{cases} 1/\lambda^2 & \text{if} \quad 0 < \lambda \leq 1 \\ 1 & \text{if} \quad \lambda \geq 1 \end{cases}.$$

then (1) and (2) are verified with $\overline{\Pi}$ *instead of* Π *and* $\overline{Q}_1, \overline{Q}_2, \overline{Q}_3$ *instead of* Q_1, Q_2, Q_3 *respectively. Given two projections* $\Pi_1, \Pi_2 : \mathbb{E}^3 \to \omega$ *, we will later write that*

$$\Pi_1 \sim \Pi_2 \quad \Leftrightarrow \quad \Pi_1 = \Pi_2 \ \text{ or } \ \Pi_1 = \overline{\Pi}_2 . \tag{27}$$

The same considerations apply to the secondary Pohlke's ellipse $\mathcal{E}_{\mathrm{S}}$ *(and to the corresponding sphere* $\widetilde{S}$ *and projection* $\widetilde{\Pi}$*) when (8)-(9) holds.* □

Remark 2.2. *Looking at (19) - (22), it is worth noting that the Pohlke's projection* Π *depends only on the quantities* γ, λ *which we have defined in (18). Taking into account (23), it is also immediate that:*

$\mathcal{E}_{\mathrm{P}}$ *remains unchanged if* $\|A_1\|$, $\|A_2\|$ *and* $A_1 \cdot A_2$ *do not vary.*

Using (25) and the expressions (28) of the lengths of the semi-axes of $\mathcal{E}_{\mathrm{P}}$*, it is possible to prove that the converse of this last statement is also true.* □

3. The Pohlke's ellipse $\mathcal{E}_{\mathrm{P}}$

As in the previous section, we suppose that OP_1, OP_2, OP_3 are not all parallel and we use a system of coordinate axes x, y, z such that ω is the plane $z = 0$ and (16) holds.

Lemma 3.1. *The lengths* σ_- , σ_+ *of the semi-axes of the Pohlke's ellipse* $\mathcal{E}_{\mathrm{P}}$ *are given by*

$$(\sigma_\pm)^2 = \frac{\|A_1\|^2 + \|A_2\|^2 \pm \sqrt{\left(\|A_1\|^2 + \|A_2\|^2\right)^2 - 4\|A_1 \wedge A_2\|^2}}{2} . \tag{28}$$

Proof. Since $\mathcal{E}_{\mathrm{P}} = \Pi(S \cap \pi)$, it is clear that $\sigma_- = \rho$ where ρ is the radius of S given by (23). From (22) we can also see that $\sigma_+ = \rho\sqrt{1 + \alpha^2 + \beta^2}$ because the direction of projection is given by the column vector

$$\overrightarrow{u} = \begin{pmatrix} -\alpha \\ -\beta \\ 1 \end{pmatrix}. \tag{29}$$

Taking account (19) and (23), we have

$$\sigma_-^2 = \frac{\|A_2\|^2}{\eta} = \|A_2\|^2 \, \frac{\lambda^2 + 1 - \sqrt{(\lambda^2+1)^2 - 4\lambda^2 \sin^2\!\gamma}}{2} . \tag{30}$$

While, by (19), (20) and (23) we obtain

$$\begin{aligned} \sigma_+^2 &= \rho^2 (1 + \alpha^2 + \beta^2) \\ &= \frac{\|A_2\|^2}{\eta} \left(\eta\lambda^2 + \eta - 1\right) = \|A_2\|^2 \left(\lambda^2 + 1 - \eta^{-1}\right) \\ &= \|A_2\|^2 \, \frac{\lambda^2 + 1 + \sqrt{(\lambda^2+1)^2 - 4\lambda^2 \sin^2\!\gamma}}{2} . \end{aligned} \tag{31}$$

Using the definitions (18) of γ and λ and noting that

$$\|A_1\| \, \|A_2\| \sin\gamma = \|A_1 \wedge A_2\|, \tag{32}$$

we obtain (28). □

Remark 3.1. *σ_-, σ_+ are also the lengths of the semi-axes of the ellipse $\mathcal{E}$ defined by the pair of conjugate semi-diameters (OA_1, OA_2).*[3] *In fact, by Apollonius's theorems on conjugate diameters, the lengths a, b of these semi-axes satisfy the system*

$$a^2 + b^2 = \|A_1\|^2 + \|A_2\|^2, \quad ab = \|A_1 \wedge A_2\|. \tag{33}$$

Thus we immediately find

$$a^2,\, b^2 = \frac{\|A_1\|^2 + \|A_2\|^2 \pm \sqrt{\left(\|A_1\|^2 + \|A_2\|^2\right)^2 - 4\|A_1 \wedge A_2\|^2}}{2}, \tag{34}$$

i.e., formula (28). □

Remark 3.2. *Noting (17), from (28) we get*

$$\sigma_-^2 + \sigma_+^2 = \|A_1\|^2 + \|A_2\|^2 = |OP_1|^2 + |OP_2|^2 + |OP_3|^2. \tag{35}$$

See also [2, Main Theorem 3.1] *for an alternative proof of (35).* □

Lemma 3.2. *If one of the segments OP_1, OP_2, OP_3 vanishes then $\mathcal{E}_{\mathrm{P}}$ is the ellipse determined by the pair of conjugate semi-diameters given by the other two segments.*

Proof. Suppose OP_3 vanishes. Then we must prove that $\mathcal{E}_{\mathrm{P}} = \mathcal{E}_{P_1,P_2}$. Namely, the Pohlke's ellipse $\mathcal{E}_{\mathrm{P}}$ is determined by the pair of conjugate semi-diameters (OP_1, OP_2). We can argue in various ways:

- Since $P_3 = O$, in (1) the direction of the projection Π is given by the segments OQ_3. By the orthogonality conditions (2) this means that $Q_1, Q_2 \in \pi$. Hence, it follows that

$$\mathcal{E}_{P_1,P_2} = \Pi(S \cap \pi) = \mathcal{E}_{\mathrm{P}}. \tag{36}$$

- Since $\mathcal{E}_{\mathrm{P}}$ and $\mathcal{E}_{P_1,P_2}$ are concentric and tangent at some point P, there exists $P', P'' \neq O$, with $OP' \parallel OP''$ and $OP' \supset OP''$, such that $\mathcal{E}_{\mathrm{P}}$ and $\mathcal{E}_{P_1,P_2}$ are determined by the pairs of conjugate semi-diameters (OP, OP') and (OP, OP'') respectively. By Apollonius's theorem on conjugate semi-diameters and Remark 3.2 we have

$$|OP|^2 + |OP'|^2 = |OP_1|^2 + |OP_2|^2 = |OP|^2 + |OP''|^2. \tag{37}$$

  This gives $|OP'| = |OP''|$ and we deduce that $\mathcal{E}_{\mathrm{P}} = \mathcal{E}_{P_1,P_2}$ because they are determined by the same pair of conjugate semi-diameters.

- Lemma 3.2 is immediate if we also consider the degenerate ellipses.[1] Indeed, if $OP_3 = O$ then $\mathcal{E}_{P_1,P_3} = \mathcal{E}_{P_1,O} = P_1P_1'$ and $\mathcal{E}_{P_2,P_3} = \mathcal{E}_{P_2,O} = P_2P_2'$, where the points P_1', P_2' are symmetric to P_1, P_2 respectively, with respect to O. It follows that $P_1, P_2 \in \mathcal{E}_{\mathrm{P}}$ and that the ellipses $\mathcal{E}_{\mathrm{P}}$ and $\mathcal{E}_{P_1,P_2}$ are tangent at P_1 and P_2. Hence $\mathcal{E}_{\mathrm{P}} = \mathcal{E}_{P_1,P_2}$. See also [6, Section 3].

Having proved Lemma 3.2, we now suppose that the segments OP_1, OP_2, OP_3 do not vanish. We begin with a special case:

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[3] Here, with a slight abuse of notation, we use A_1, A_2 to indicate two points with the same coordinates of the rows A_1, A_2 of the matrix A defined in (17).

Lemma 3.3. *Let us suppose that*

$$U_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} h \\ k \\ 0 \end{pmatrix}, \tag{38}$$

with h, k not both zero (i.e., $U_3 \neq O$). Then the semi-axes of the Pohlke's ellipse $\mathcal{E}_{\mathrm{P}}(O, U_1, U_2, U_3)$ are the segments $O\Sigma_-$ and $O\Sigma_+$ with

$$\Sigma_- = \frac{\pm 1}{\sqrt{h^2 + k^2}} \begin{pmatrix} k \\ -h \\ 0 \end{pmatrix}, \qquad \Sigma_+ = \pm\sqrt{\frac{1 + h^2 + k^2}{h^2 + k^2}} \begin{pmatrix} h \\ k \\ 0 \end{pmatrix}. \tag{39}$$

Proof. According to (16), (17) we set

$$A = \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 0 \end{pmatrix} \tag{40}$$

and then we follow the scheme from (18) to (23). We have

$$\cos\gamma = \frac{hk}{\sqrt{1 + h^2}\,\sqrt{1 + k^2}}\,, \quad \lambda = \frac{\sqrt{1 + h^2}}{\sqrt{1 + k^2}}\,. \tag{41}$$

From this we get $\eta = 1 + k^2$, $\rho = \|A_2\|\,\eta^{-1/2} = 1$ and

$$(\alpha, \beta) = \pm\big(|h|, \operatorname{sgn}(hk)|k|\big) = \pm(h, k). \tag{42}$$

It follows that the lengths of the semi-axes are

$$\sigma_- = 1 \quad \text{and} \quad \sigma_+ = \sqrt{1 + h^2 + k^2}. \tag{43}$$

Moreover, the direction of projection onto the image plane ω is given by the nonzero vector $\overrightarrow{v} = \begin{pmatrix} -h \\ -k \\ 1 \end{pmatrix}$ or $\begin{pmatrix} h \\ k \\ 1 \end{pmatrix}$. This means that

$$O\Sigma_- \parallel \begin{pmatrix} h \\ -k \\ 0 \end{pmatrix}, \quad O\Sigma_+ \parallel \begin{pmatrix} h \\ k \\ 0 \end{pmatrix} \tag{44}$$

and then we can easily derive the expressions (39) for Σ_- and Σ_+. □

Remark 3.3. *It is easy to find the Pohlke's projection corresponding to U_1, U_2, U_3 directly. Indeed, in view of (38), $\mathcal{E}_{U_1,U_2}$ is a circle with center O and radius $\rho = 1$. Hence, $\mathcal{E}_{\mathrm{P}}(O, U_1, U_2, U_3)$ must have semi-minor axis $\sigma_- = 1$. This means that S has radius $\rho = 1$ and that the conditions (1), (2) are satisfied (with $P_i = U_i$, $1 \leq i \leq 3$) taking $Q_1 = U_1$, $Q_2 = U_2$,*

$$Q_3 = \pm \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{45}$$

and the direction of the projection Π *parallel to the segment Q_3U_3, i.e., the vector $\overrightarrow{v}$ above. See* [6, Section 4] *for more details.* □

We are now in position to obtain the expressions of the conjugate semi-diameters of $\mathcal{E}_{\mathrm{P}}$ in the general case:

Lemma 3.4. *Suppose* $OP_1 \nparallel OP_2$ *and*

$$\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2}, \tag{46}$$

with h, k *not both zero* (*i.e.*, $OP_3 \neq O$). *Then the segments* OV, OW *with*

$$\begin{aligned} \overrightarrow{OV} &= \pm \frac{k\,\overrightarrow{OP_1} - h\,\overrightarrow{OP_2}}{\sqrt{h^2+k^2}} \quad \text{and} \\ \overrightarrow{OW} &= \pm \sqrt{\frac{1+h^2+k^2}{h^2+k^2}} \left(h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2} \right) \end{aligned} \tag{47}$$

are conjugate semi-diameters of the Pohlke's ellipse $\mathcal{E}_{\mathrm{P}}$.

Proof. Noting that $OV \nparallel OW$, it is enough to show that $\mathcal{E}_{\mathrm{P}}(O, P_1, P_2, P_3)$ coincides with the Pohlke's ellipse $\mathcal{E}_{\mathrm{P}}(O, V, W, O)$, where the third segment vanishes. Indeed, by Lemma 3.2, OV and OW are conjugate semi-diameters of $\mathcal{E}_{\mathrm{P}}(O, V, W, O)$. To prove this fact, we consider the matrix $\mathcal{A}$ given by the coordinates of the points V, W and O. Namely, we set

$$\mathcal{A} = \begin{pmatrix} \frac{1}{\sqrt{h^2+k^2}}(kx_1 - hx_2) & \sqrt{\frac{1+h^2+k^2}{h^2+k^2}}(hx_1 + kx_2) & 0 \\ \frac{1}{\sqrt{h^2+k^2}}(ky_1 - hy_2) & \sqrt{\frac{1+h^2+k^2}{h^2+k^2}}(hy_1 + ky_2) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{48}$$

where, for simplicity, in (47) we always choose the sign " + ". Then we evaluate the norms and the scalar product of the rows $\mathcal{A}_1$, $\mathcal{A}_2$ of $\mathcal{A}$. By (46) we have $x_3 = hx_1 + kx_2$ and $y_3 = hy_1 + ky_2$. Thus, we obtain:

$$\begin{aligned} \|\mathcal{A}_1\|^2 &= \frac{(kx_1 - hx_2)^2}{h^2+k^2} + \frac{1+h^2+k^2}{h^2+k^2}(hx_1 + kx_2)^2 \\ &= \frac{(hx_1 + kx_2)^2 + (kx_1 - hx_2)^2}{h^2+k^2} + (hx_1 + kx_2)^2 \\ &= x_1^2 + x_2^2 + x_3^2 = \|A_1\|^2, \end{aligned} \tag{49}$$

and in the same way we can show that $\|\mathcal{A}_2\|^2 = \|A_2\|^2$.

Further, we consider the scalar product $\mathcal{A}_1 \cdot \mathcal{A}_2$. We have:

$$\begin{aligned} \mathcal{A}_1 \cdot \mathcal{A}_2 &= \frac{(kx_1 - hx_2)(ky_1 - hy_2)}{h^2+k^2} + \frac{1+h^2+k^2}{h^2+k^2}(hx_1 + kx_2)(hy_1 + ky_2) \\ &= \frac{(hx_1 + kx_2)(hy_1 + ky_2) + (kx_1 - hx_2)(ky_1 - hy_2)}{h^2+k^2} \\ &\quad + (hx_1 + kx_2)(hy_1 + ky_2) \\ &= x_1 y_1 + x_2 y_2 + x_3 y_3 = A_1 \cdot A_2. \end{aligned}$$

In conclusion, we find that

$$\|\mathcal{A}_1\| = \|A_1\|, \quad \|\mathcal{A}_2\| = \|A_2\| \quad \text{and} \quad \mathcal{A}_1 \cdot \mathcal{A}_2 = A_1 \cdot A_2\,. \tag{50}$$

By Remark 2.2, this means that $\mathcal{E}_{\mathrm{P}}(O, V, W, O) = \mathcal{E}_{\mathrm{P}}(OP_1, P_2, P_3)$. □

Summing up from Lemmas 3.2, 3.3 and 3.4, we get:

Theorem 3.1. *Let us suppose that $OP_1 \nparallel OP_2$. If $OP_3 = O$ then the segments OP_1, OP_2 are conjugate semi-diameters of the Pohlke's ellipse $\mathcal{E}_{\mathtt{P}}$. Conversely, if $OP_3 \neq O$ then a pair of conjugate semi-diameters is given by the segments OV, OW with*

$$\overrightarrow{OV} = \pm \frac{k\,\overrightarrow{OP_1} - h\,\overrightarrow{OP_2}}{\sqrt{h^2+k^2}} \quad \text{and} \quad \overrightarrow{OW} = \pm \sqrt{\frac{1+h^2+k^2}{h^2+k^2}}\,\overrightarrow{OP_3}, \tag{51}$$

where the coefficients h, k are such that

$$\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2}. \tag{52}$$

With U_1, U_2, U_3 as in (38) we also have:

Lemma 3.5. *Let $\Phi : \omega \to \omega$ be the an affine transformation and let us suppose that*

$$OP_1 = \Phi(OU_1), \quad OP_2 = \Phi(OU_2), \quad OP_3 = \Phi(OU_3). \tag{53}$$

Then $\Phi(\mathcal{E}_{\mathtt{P}}(O, U_1, U_2, U_3)) = \mathcal{E}_{\mathtt{P}}(O, P_1, P_2, P_3)$.

Proof. From (53) it is clear that $OU_1 \nparallel OU_2 \Rightarrow OP_1 \nparallel OP_2$ and that

$$\overrightarrow{OU_3} = h\,\overrightarrow{OU_1} + k\,\overrightarrow{OU_2} \quad \Rightarrow \quad \overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2}. \tag{54}$$

If $U_3 = O$ then $P_3 = O$ and by the first part of Theorem 3.1, we know that

$$\mathcal{E}_{\mathtt{P}}(O, U_1, U_2, O) \quad \text{and} \quad \mathcal{E}_{\mathtt{P}}(O, P_1, P_2, O)$$

are determined by the pairs of conjugate semi-diameters (OU_1, OU_2) and (OP_1, OP_2) respectively. Since $OP_1 = \Phi(OU_1)$ and $OP_2 = \Phi(OU_2)$ it follows that

$$\Phi(\mathcal{E}_{\mathtt{P}}(O, U_1, U_2, O)) = \mathcal{E}_{\mathtt{P}}(O, P_1, P_2, O). \tag{55}$$

Conversely, by the second part of Theorem 3.1, if $U_3 \neq O$ then the ellipses $\mathcal{E}_{\mathtt{P}}(O, U_1, U_2, U_3)$ and $\mathcal{E}_{\mathtt{P}}(O, P_1, P_2, P_3)$ are determined by the pairs of conjugate semi-diameters $(O\Sigma_-, O\Sigma_+)$ (given by (39)) and (OV, OW) respectively. Since

$$\Phi(\overrightarrow{O\Sigma_-}) = \pm\overrightarrow{OV} \quad \text{and} \quad \Phi(\overrightarrow{O\Sigma_+}) = \pm\overrightarrow{OW}, \tag{56}$$

we come to the same conclusion. □

More generally, applying Lemma 3.5, we can easily prove the following:

Theorem 3.2. *Let $\Psi : \omega \to \omega$ be any affine transformation. Suppose the segments OP_1, OP_2, OP_3 are not all parallel and let $\mathcal{E}_{\mathtt{P}}$ be the corresponding Pohlke's ellipse. Then $\Psi(\mathcal{E}_{\mathtt{P}})$ is the Pohlke's ellipse corresponding to the triad of segments $\Psi(OP_1)$, $\Psi(OP_2)$, $\Psi(OP_3)$.*

Remark 3.4. *Suppose OP_1, OP_2, OP_3 are not all parallel and do not vanish. Let T_{ij} $(i \neq j)$ be a point of contact of $\mathcal{E}_{\mathtt{P}}(O, P_1, P_2, P_3)$ with $\mathcal{E}_{P_i,P_j}$. Let t_{ij} be the common tangent line at T_{ij}. Then we can easily show that*

$$t_{ij} \parallel OP_k \quad (k \neq i, j). \tag{57}$$

Indeed, by Lemma 3.3, if $OP_i \nparallel OP_j$ it is sufficient to observe that the statement is true for the ellipses $\mathcal{E}_{\mathtt{P}}(O, U_1, U_2, U_3)$ and $\mathcal{E}_{U_1,U_2}$. Conversely, if $OP_i \parallel OP_j$, taking $h \neq 0$ and $k = 0$ in (38), we note that the conclusion is true for $\mathcal{E}_{\mathtt{P}}(O, U_1, U_2, U_3)$ and the degenerate ellipses $\mathcal{E}_{U_1,U_3}$, with $U_3 \neq O$

and $OU_1 \parallel OU_3$. [1] *This result was first derived in* [3, Theorem 2] *through synthetic methods.* □

4. The secondary Pohlke's ellipse $\mathcal{E}_{\mathrm{S}}$

In this section we suppose the segments OP_1, OP_2, OP_3 are non-parallel (i.e., $OP_i \nparallel OP_j$ if $i \neq j$) and

$$\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2} \tag{58}$$

with $h, k \neq 0$ such that

$$g(h,k) \stackrel{\text{def}}{=} h^4 + k^4 - 2h^2k^2 - 2h^2 - 2k^2 + 1 > 0. \tag{59}$$

By Theorem 1.1 there exists a unique secondary Pohlke's ellipse $\mathcal{E}_{\mathrm{S}}$. [4]

As in the previous section we use a system of coordinate axes x, y, z such that ω is the plane $z = 0$ and (16) holds. Also we first consider the triad of segments OU_1, OU_2, OU_3 where

$$U_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad U_3 = \begin{pmatrix} h \\ k \\ 0 \end{pmatrix} \quad \text{with} \quad h, k \neq 0 \tag{60}$$

as above. Then, since OU_1, OU_2, OU_3 are non-parallel and

$$\overrightarrow{OU_3} = h\,\overrightarrow{OU_1} + k\,\overrightarrow{OU_2}, \tag{61}$$

the secondary Pohlke's ellipse $\mathcal{E}_{\mathrm{S}}(O, U_1, U_2, U_3)$ exists and it is unique. More precisely, from [6, Section 4], we know that the conditions (4), (5) and (6) (with $P_i = U_i$, for $1 \leq i \leq 3$) are verified by taking: $\widetilde{S}$ the sphere with center O and radius $\rho = 1$, the points

$$R_1 = U_1, \quad R_2 = U_2 \quad \text{and} \quad R_3 = \frac{1}{h^2 - k^2 + 1} \begin{pmatrix} 2h \\ 0 \\ \pm\sqrt{g(h,k)} \end{pmatrix}, \tag{62}$$

where $g(h,k)$ is the function defined in (59). [5] See formula (90) of [6]. This means that the direction of the projection $\widetilde{\Pi}$ is given by the vector $\overrightarrow{R_3U_3}$. From these facts it follows that:

Lemma 4.1. *Suppose (59), (60) hold. Then the semi-axes of the secondary Pohlke's ellipse* $\mathcal{E}_{\mathrm{S}}(O, U_1, U_2, U_3)$ *are represented by the segments* $O\widetilde{\Sigma}_-$ *and* $O\widetilde{\Sigma}_+$ *with*

$$\widetilde{\Sigma}_- = \frac{\pm 1}{\sqrt{H^2 + K^2}} \begin{pmatrix} K \\ -H \\ 0 \end{pmatrix}, \qquad \widetilde{\Sigma}_+ = \pm\sqrt{\frac{g + H^2 + K^2}{g(H^2 + K^2)}} \begin{pmatrix} H \\ K \\ 0 \end{pmatrix} \tag{63}$$

[4] Condition (58)-(59) is clearly equivalent to (8)-(9). But (58)-(59) allows us to obtain slight simpler expressions.

[5] Note that condition (59) $\Rightarrow h^2 - k^2 \neq \pm 1$. In fact, since $g(h,k) = (h^2 - k^2)^2 - 2h^2 - 2k^2 + 1$, we get

$$h^2 - k^2 = \pm 1 \quad \Rightarrow \quad g(h,k) = 2(1 - h^2 - k^2) = \begin{cases} -4h^2 & \text{if} \quad h^2 - k^2 = -1 \\ -4k^2 & \text{if} \quad h^2 - k^2 = 1 \end{cases}.$$

where $g = g(h,k)$ *and*

$$H \overset{def}{=} h(h^2 - k^2 - 1), \quad K \overset{def}{=} k(h^2 - k^2 + 1). \tag{64}$$

Proof. From (62) we have

$$\overrightarrow{R_3U_3} = \frac{1}{h^2 - k^2 + 1} \begin{pmatrix} h(h^2 - k^2 - 1) \\ k(h^2 - k^2 + 1) \\ \mp\sqrt{g(h,k)} \end{pmatrix}. \tag{65}$$

Thus multiplying the right hand side of (65) by the factor $\frac{h^2-k^2+1}{\sqrt{g(h,k)}}$ we see that the direction of projection is given by the vector

$$\overrightarrow{w} = \begin{pmatrix} -H/\sqrt{g} \\ -K/\sqrt{g} \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} H/\sqrt{g} \\ K/\sqrt{g} \\ 1 \end{pmatrix}, \tag{66}$$

where the terms H, K are defined as in (64). Furthermore $H, K \neq 0$ because $h, k \neq 0$ and condition (59) holds. [5] Then, taking into account that the sphere $\widetilde{S}$ has center O and radius $\rho = 1$ we easily get (63). □

Corollary 4.1. *Suppose (59), (60) hold. Then*

$$\text{area}\big(\mathcal{E}_{\mathtt{P}}(O, U_1, U_2, U_3)\big) < \text{area}\big(\mathcal{E}_{\mathtt{S}}(O, U_1, U_2, U_3)\big). \tag{67}$$

Proof. From the expressions (39) and (63) we have $|O\Sigma_-| = |O\widetilde{\Sigma}_-| = 1$. Thus it is enough to prove the inequality $|O\Sigma_+|^2 < |O\widetilde{\Sigma}_+|^2$, that is

$$1 + h^2 + k^2 < \frac{g + H^2 + K^2}{g}. \tag{68}$$

Since we know that $g > 0$, (68) is equivalent to $(h^2 + k^2)\, g < H^2 + K^2$. Introducing the expressions (59) and (64), with elementary calculations the last inequality reduces to

$$0 < h^2k^2, \tag{69}$$

which is clearly verified because we are assuming $h, k \neq 0$. □

We can now give the expressions of a pair of conjugate semi-diameters of the secondary Pohlke's ellipse $\mathcal{E}_{\mathtt{S}}(O, P_1, P_2, P_3)$. Indeed, with U_1, U_2, U_3 as in (60), we have:

Lemma 4.2. *Suppose the segments* OP_1, OP_2, OP_3 *are non-parallel and condition (58)-(59) (or (8)-(9)) holds. Let* $\Phi : \omega \to \omega$ *be the affine transformation such that* $OP_1 = \Phi(OU_1)$, $OP_2 = \Phi(OU_2)$. *Then*

$$\Phi(\mathcal{E}_{\mathtt{S}}(O, U_1, U_2, U_3)) = \mathcal{E}_{\mathtt{S}}(O, P_1, P_2, P_3). \tag{70}$$

In particular the segments $O\widetilde{V}$ *and* $O\widetilde{W}$, *with*

$$\begin{aligned} \overrightarrow{O\widetilde{V}} &= \pm \frac{K\,\overrightarrow{OP_1} - H\,\overrightarrow{OP_2}}{\sqrt{H^2 + K^2}}, \\ \overrightarrow{O\widetilde{W}} &= \pm \sqrt{\frac{g + H^2 + K^2}{g(H^2 + K^2)}}\,\Big(H\,\overrightarrow{OP_1} + K\,\overrightarrow{OP_2}\Big), \end{aligned} \tag{71}$$

are conjugate semi-diameters of $\mathcal{E}_{\mathtt{S}}(O, P_1, P_2, P_3)$.

Proof. In view of Pohlke's theorem and Theorem 1.1, there are exactly two distinct ellipses with center O and circumscribing $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$. Namely, the Pohlke's ellipse $\mathcal{E}_{\mathtt{P}}(O,P_1,P_2,P_3)$ and the secondary Pohlke's ellipse $\mathcal{E}_{\mathtt{S}}(O,P_1,P_2,P_3)$. Noting that $\Phi(OU_3)=OP_3$, we have

$$\Phi(\mathcal{E}_{U_1,U_2})=\mathcal{E}_{P_1,P_2},\quad \Phi(\mathcal{E}_{U_2,U_3})=\mathcal{E}_{P_2,P_3},\quad \Phi(\mathcal{E}_{U_3,U_1})=\mathcal{E}_{P_3,P_1}. \tag{72}$$

Since the ellipse $\mathcal{E}_{\mathtt{S}}(O,U_1,U_2,U_3)$ circumscribes $\mathcal{E}_{U_1,U_2}$, $\mathcal{E}_{U_2,U_3}$ and $\mathcal{E}_{U_3,U_1}$, we deduce that

$$\Phi(\mathcal{E}_{\mathtt{S}}(O,U_1,U_2,U_3))\quad \text{circumscribes}\quad \mathcal{E}_{P_1,P_2},\,\mathcal{E}_{P_2,P_3},\,\mathcal{E}_{P_3,P_1}. \tag{73}$$

By Lemma 3.5 we know that $\Phi(\mathcal{E}_{\mathtt{P}}(O,U_1,U_2,U_3))=\mathcal{E}_{\mathtt{P}}(O,P_1,P_2,P_3)$, thus we must conclude that

$$\Phi(\mathcal{E}_{\mathtt{S}}(O,U_1,U_2,U_3))=\mathcal{E}_{\mathtt{S}}(O,P_1,P_2,P_3) \tag{74}$$

because $\mathcal{E}_{\mathtt{S}}(O,U_1,U_2,U_3)\neq\mathcal{E}_{\mathtt{P}}(O,U_1,U_2,U_3)$.
Finally, taking account Lemma 4.1, we see that the segments $\Phi(O\widetilde{\Sigma}_-)$ and $\Phi(O\widetilde{\Sigma}_+)$ are conjugate semi-diameters of $\Phi(\mathcal{E}_{\mathtt{S}}(O,U_1,U_2,U_3))$, hence the segments $O\widetilde{V}, O\widetilde{W}$ given by (71) are conjugate semi-diameters of the secondary Pohlke's ellipse $\mathcal{E}_{\mathtt{S}}(O,P_1,P_2,P_3)$. □

From Corollary 4.1 and Lemma 4.2 it is now clear that:

Corollary 4.2. *Suppose the segments OP_1,OP_2,OP_3 are non-parallel and condition (58)-(59) (i.e., (8)-(9)) holds. Then*

$$\operatorname{area}\big(\mathcal{E}_{\mathtt{P}}(O,P_1,P_2,P_3)\big)<\operatorname{area}\big(\mathcal{E}_{\mathtt{S}}(O,P_1,P_2,P_3)\big). \tag{75}$$

More generally, if $\Psi:\omega\to\omega$ is any affine transformation of the plane ω, applying the previous results we can easily prove the following:

Theorem 4.1. *Suppose the segments OP_1,OP_2,OP_3 are non-parallel and condition (8)-(9) holds. Let $\mathcal{E}_{\mathtt{S}}$ be the secondary Pohlke's ellipse of the triad OP_1,OP_2,OP_3. Then $\Psi(\mathcal{E}_{\mathtt{S}})$ is the secondary Pohlke's ellipse of the triad of segments $\Psi(OP_1)$, $\Psi(OP_2)$, $\Psi(OP_3)$.*

5. A determination of the secondary Pohlke's projection

In this section we give formulae for determining the secondary Pohlke's projection of a triad of non-parallel segments OP_1,OP_2,OP_3 satisfying the condition (8)-(9). That is a parallel projection $\widetilde{\Pi}:\mathbb{E}^3\to\omega$, a sphere $\widetilde{S}$ with center O and three points $R_1,R_2,R_3\in\widetilde{S}$ such that (4), (5), (6) hold.

We already know that $\widetilde{S}$ is unique, $\widetilde{\Pi}$ is unique up to symmetry with respect to ω, and that the set $\{R_1,R_2,R_3\}$ is determined up to symmetry with respect to ω and up to symmetry with respect to a plane $\widetilde{\pi}$ through O and perpendicular to the direction of projection. See Remark 2.1 and [6].

To begin with, we note the following:

Claim 5.1. *Let $\widetilde{\Pi}:\mathbb{E}^3\to\omega$ be a secondary Pohlke's projection for OP_1, OP_2,OP_3 and suppose the nonzero vector $\overrightarrow{w}$ represents the direction of this projection. Then the following hold:*

(*a*) $OR_i, OR_i' \nparallel \overrightarrow{w}$ $(1\leq i\leq 3)$.

(*b*) *If the vector* $\overrightarrow{w}$ *is known, then the points* $R_1, R_2, R_3, R'_1, R'_2, R'_3$ *can be recursively computed from any of them. For example, if* R_3 *is given then we immediately have:*

$$\overrightarrow{OR_2} = \overrightarrow{OP_2} - \frac{\overrightarrow{OR_3} \cdot \overrightarrow{OP_2}}{\overrightarrow{OR_3} \cdot \overrightarrow{w}}\, \overrightarrow{w}, \quad \overrightarrow{OR'_1} = \overrightarrow{OP_1} - \frac{\overrightarrow{OR_3} \cdot \overrightarrow{OP_1}}{\overrightarrow{OR_3} \cdot \overrightarrow{w}}\, \overrightarrow{w}. \tag{76}$$

Proof. (*a*) It follows from condition (6). Indeed, if $OR_i \perp \overrightarrow{w}$, or if $OR'_i \perp \overrightarrow{w}$, then $R_i = R'_i \in \widetilde{\pi}$ where $\widetilde{\pi}$ is the plane through O and perpendicular to $\overrightarrow{w}$. Thus (6) fails.

(*b*) By condition (4) we have $\widetilde{\Pi}(R_2) = P_2$, thus $\overrightarrow{OR_2} = \overrightarrow{OP_2} + t\overrightarrow{w}$ for some $t \in \mathbb{R}$. By (5) we also know that $OR_2 \perp OR_3$. So, taking account that $\overrightarrow{OR_3} \cdot \overrightarrow{w} \neq 0$, we obtain

$$t = -\frac{\overrightarrow{OR_3} \cdot \overrightarrow{OP_2}}{\overrightarrow{OR_3} \cdot \overrightarrow{w}}. \tag{77}$$

This gives the first equality of (76). Noting that $\widetilde{\Pi}(R'_1) = P_1$ and $OR_3 \perp OR'_1$, in the same way we can derive the second equality. To conclude it is enough to consider also the points R'_2 and R'_3, because from condition (5) we get a cyclic relation of orthogonality:

$$\begin{aligned} &OR_1 \perp OR_2,\ OR_2 \perp OR_3,\ OR_3 \perp OR'_1, \\ &\qquad OR'_1 \perp OR'_2,\ OR'_2 \perp OR'_3,\ OR'_3 \perp OR_1. \end{aligned} \tag{78}$$

So we can start from any point of the set $\{R_1, R_2, R_3, R'_1, R'_2, R'_3\}$. □

Next, suppose that the segments OP_1, OP_2, OP_3 are non-parallel and that the condition (58)-(59) (i.e., (8)-(9)) is true. By Theorem 2.1 of [6] there exist a sphere $\widetilde{S}$ with center O, three point $R_1, R_2, R_3 \in \widetilde{S}$ and a parallel projection $\widetilde{\Pi} : \mathbb{E}^3 \to \omega$ such that the conditions (4), (5), (6) hold. To determine R_1, R_2, R_3 and $\widetilde{\Pi}$, we begin by observing that setting

$$\overrightarrow{OX_3} = \frac{H}{\sqrt{g}}\overrightarrow{OP_1} + \frac{K}{\sqrt{g}}\overrightarrow{OP_2}, \tag{79}$$

we have

$$\mathcal{E}_{\mathtt{S}}(O, P_1, P_2, P_3) = \mathcal{E}_{\mathtt{P}}(O, P_1, P_2, X_3). \tag{80}$$

Indeed, by Lemma 3.4 the segments $O\widehat{V}$ and $O\widehat{W}$, with

$$\overrightarrow{O\widehat{V}} = \pm\frac{\frac{K}{\sqrt{g}}\overrightarrow{OP_1} - \frac{H}{\sqrt{g}}\overrightarrow{OP_2}}{\sqrt{\frac{H^2}{g} + \frac{K^2}{g}}} \qquad \text{and} \tag{81}$$

$$\overrightarrow{O\widehat{W}} = \pm\sqrt{\frac{1 + \frac{H^2}{g} + \frac{K^2}{g}}{\frac{H^2}{g} + \frac{K^2}{g}}}\left(\frac{H}{\sqrt{g}}\overrightarrow{OP_1} + \frac{K}{\sqrt{g}}\overrightarrow{OP_2}\right), \tag{82}$$

are conjugate semi-diameters of the Pohlke's ellipse $\mathcal{E}_{\mathtt{P}}(O, P_1, P_2, X_3)$. Noting the expressions (71) of Lemma 4.2, it is clear $O\widehat{V}$, $O\widehat{W}$ coincide with the conjugate semi-diameters $O\widetilde{V}, O\widetilde{W}$ respectively of the secondary Pohlke's ellipse $\mathcal{E}_{\mathtt{S}}(O, P_1, P_2, P_3)$. Thus (80) holds.

Thanks to the considerations made in Remark 2.1, this implies that the secondary Pohlke's projection $\widetilde{\Pi}$ corresponding to the triad of segments OP_1, OP_2, OP_3 and the Pohlke's projection of the triad OP_1, OP_2, OX_3 are equal. More precisely, taking account the conditions (1) and (2), let us denote with $\widehat{S}$ a sphere centered at O, with $\widehat{Q}_1, \widehat{Q}_2, \widehat{Q}_3$ three points of $\widehat{S}$ and with $\widehat{\Pi} : \mathbb{E}^3 \to \omega$ a parallel projection such that:

$$\widehat{\Pi}(O\widehat{Q}_1) = OP_1, \quad \widehat{\Pi}(O\widehat{Q}_2) = OP_2 \quad \text{and} \quad \widehat{\Pi}(O\widehat{Q}_3) = OX_3, \tag{83}$$

$$O\widehat{Q}_1 \perp O\widehat{Q}_2, \quad O\widehat{Q}_2 \perp O\widehat{Q}_3, \quad O\widehat{Q}_3 \perp O\widehat{Q}_1. \tag{84}$$

Then, by Remark 2.1, it follows that

$$\widetilde{S} = \widehat{S} \quad \text{and} \quad \widetilde{\Pi} \sim \widehat{\Pi}. \tag{85}$$

For our purposes the projection $\widetilde{\Pi}$ and its symmetric with respect to the plane ω are equivalent, thus we can take $\widetilde{\Pi} = \widehat{\Pi}$. Then, to fulfill the conditions (4), (5) and (6), we only need only to select appropriately the points $R_1, R_2, R_3 \in \widehat{S}$. More precisely,

$$R_i = \widehat{Q}_i \quad \text{or} \quad R_i = \widehat{Q}_i{}' \quad (1 \le i \le 2)\,[6] \tag{86}$$

and then $R_3 \in \widehat{S}$ such that

$$\widehat{\Pi}(R_3) = P_3 \quad \text{and} \quad OR_3 \perp OR_1'. \tag{87}$$

Thanks to the symmetry with respect to the plane $\widehat{\pi}$, it is indifferent to start with $R_1 = \widehat{Q}_1$ or $R_1 = \widehat{Q}_1{}'$. If, for instance, we start with $R_1 = \widehat{Q}_1$ then we must take

$$R_2 = \widehat{Q}_2, \tag{88}$$

because $O\widehat{Q}_1 \not\perp O\widehat{Q}_2{}'$.[7] After selecting R_2, the point R_3 can be obtained by applying Claim 5.1. Namely, we must have

$$\overrightarrow{OR_3} \stackrel{\text{def}}{=} \overrightarrow{OP_3} - \frac{\overrightarrow{OR_2} \cdot \overrightarrow{OP_3}}{\overrightarrow{OR_2} \cdot \overrightarrow{w}}\, \overrightarrow{w}, \tag{89}$$

where $\overrightarrow{w}$ is any nonzero vector representing the direction of the secondary Pohlke's projection $\widetilde{\Pi}$, i.e., the direction of the projection $\widehat{\Pi}$.

5.1. **Reference tetrahedron and direction of projection.** Summarizing up we give now a procedure for determining the points R_1, R_2, R_3 and the direction of the secondary Pohlke's projection. As for Pohlke's projection, we use a system of coordinate axes x, y, z such that ω is the plane

[6] Because $\widehat{\Pi}(\widehat{Q}_i) = \widehat{\Pi}(\widehat{Q}_i{}') = P_i$, for $1 \le i \le 2$. According to the previous notation, $\widehat{Q}_i{}'$ is symmetric to $\widehat{Q}_i$ with respect to the plane $\widehat{\pi}$ through O and perpendicular to the direction of the projection $\widehat{\Pi}$.

[7] Indeed, $O\widehat{Q}_1 \perp O\widehat{Q}_2 \wedge O\widehat{Q}_1 \perp O\widehat{Q}_2{}' \Rightarrow \widehat{Q}_1 \in \widehat{\pi} \vee \widehat{Q}_2 \in \widehat{\pi}$. But this cannot happen because, by (85), we already know that $\widehat{\Pi} = \widetilde{\Pi}$ is as secondary Pohlke's projection for the triad of segments OP_1, OP_2, OP_3.

$z = 0$ and (16) holds. We suppose that OP_1, OP_2, OP_3 are non-parallel and that condition (58)-(59) is true. Then we consider the matrix

$$\widehat{A} = \begin{pmatrix} x_1 & x_2 & \widehat{x}_3 \\ y_1 & y_2 & \widehat{y}_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \widehat{A}_1 \\ \widehat{A}_2 \\ 0 \end{pmatrix}, \tag{90}$$

where

$$\widehat{x}_3 = \frac{H}{\sqrt{g}}\, x_1 + \frac{K}{\sqrt{g}}\, x_2\,, \quad \widehat{y}_3 = \frac{H}{\sqrt{g}}\, y_1 + \frac{K}{\sqrt{g}}\, y_2 \tag{91}$$

and $H = h(h^2 - k^2 - 1)$, $K = k(h^2 - k^2 + 1)$ are the terms introduced in (64) with h, k as in (58). Having defined the matrix $\widehat{A}$, we continue by following the formulae (3.6), (3.10), (3.21), (3.22) of [4]. We define the quantities:

$$\hat{\gamma} = \arccos\left(\frac{\widehat{A}_1 \cdot \widehat{A}_2}{\|\widehat{A}_1\|\, \|\widehat{A}_2\|}\right), \quad \hat{\lambda} = \frac{\|\widehat{A}_1\|}{\|\widehat{A}_2\|}\,, \tag{92}$$

$$\hat{\eta} = \frac{\hat{\lambda}^2 + 1 + \sqrt{(\hat{\lambda}^2 + 1)^2 - 4\hat{\lambda}^2 \sin^2\hat{\gamma}}}{2\,\hat{\lambda}^2 \sin^2\hat{\gamma}}\,, \tag{93}$$

$$\hat{\nu} \;=\; \pm\,\hat{\rho} \quad \text{with} \quad \hat{\rho} = \frac{\|\widehat{A}_1\|}{\hat{\lambda}\sqrt{\hat{\eta}}} = \frac{\|\widehat{A}_2\|}{\sqrt{\hat{\eta}}}\,, \tag{94}$$

and, finally,

$$\left(\,\hat{\alpha}, \hat{\beta}\,\right) = \pm\left(\sqrt{\hat{\eta}\,\hat{\lambda}^2 - 1}\;,\; \operatorname{sgn}(\cos\hat{\gamma})\sqrt{\hat{\eta} - 1}\;\right), \tag{95}$$

where $t \mapsto \operatorname{sgn}(t)$ is the "signum" function introduced in (21). Then, by the results of [4, Section 4], the coordinates of the points $\widehat{Q}_1$, $\widehat{Q}_2$, $\widehat{Q}_3$ satisfying (83), (84) are the columns $\widehat{B}^1$, $\widehat{B}^2$, $\widehat{B}^3$ respectively of the matrix

$$\begin{aligned} \widehat{B} = \frac{1}{1 + \hat{\alpha}^2 + \hat{\beta}^2} & \begin{pmatrix} 1 + \hat{\beta}^2 & -\hat{\alpha}\,\hat{\beta} & -\hat{\alpha} \\ -\hat{\alpha}\,\hat{\beta} & 1 + \hat{\alpha}^2 & -\hat{\beta} \\ \hat{\alpha} & \hat{\beta} & 1 \end{pmatrix} \\ & \cdot \begin{pmatrix} x_1 & x_2 & \widehat{x}_3 \\ y_1 & y_2 & \widehat{y}_3 \\ \frac{x_2\,\widehat{y}_3 - y_2\,\widehat{x}_3}{\hat{\nu}} & \frac{y_1\,\widehat{x}_3 - x_1\,\widehat{y}_3}{\hat{\nu}} & \frac{x_1\,y_2 - y_1\,x_2}{\hat{\nu}} \end{pmatrix}. \end{aligned} \tag{96}$$

The direction of the projection $\widehat{\Pi} : \mathbb{E}^3 \to \omega$ is determined by the vector

$$\overrightarrow{w} \;=\; \begin{pmatrix} -\hat{\alpha} \\ -\hat{\beta} \\ 1 \end{pmatrix}. \tag{97}$$

Recalling the arguments from (86) to (89), it is now sufficient to modify the third column of $\widehat{B} = (\widehat{B}^1, \widehat{B}^2, \widehat{B}^3)$. More precisely, we define the matrix $\widetilde{B} = (\widetilde{B}^1, \widetilde{B}^2, \widetilde{B}^3)$ by setting

$$\widetilde{B}^1 = \widehat{B}^1, \quad \widetilde{B}^2 = \widehat{B}^2 \quad \text{and} \quad \widetilde{B}^3 = P_3 - \frac{\widehat{B}^2 \cdot P_3}{\widehat{B}^2 \cdot \overrightarrow{w}}\,\overrightarrow{w}. \tag{98}$$

The coordinates of the points R_1, R_2, R_3 are then the columns $\widetilde{B}^1, \widetilde{B}^2, \widetilde{B}^3$ respectively and the direction of the secondary Pohlke's projection $\widetilde{\Pi}$ is represented by $\overrightarrow{w}$ defined in (97), so we have

$$\widetilde{\Pi}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+\hat{\alpha}z \\ y+\hat{\beta}z \\ 0 \end{pmatrix}. \tag{99}$$

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