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A MODEL FOR COMPOSITE BRICKWORK-LIKE MATERIALS BASED ON DISCRETE ELEMENT PROCEDURE: SENSITIVITY TO DIFFERENT STAGGERED TIERS

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A discrete element model is proposed to investigate the behaviour of a 2D composite brickwork system with regular arrangement between blocks and different staggered tiers. The hypothesis of rigid blocks connected by interfaces (mortar thin joints) between blocks is assumed. In other words, masonry is considered as a "skeleton" in which the interactions between rigid blocks are represented by forces and moments which depend on their relative displacements and rotations. The hypothesis of the structural skeleton is representative of historical masonry; in fact, blocks are generally much stiffer than mortar and mortar joints show a very small thickness if compared with the sizes of the blocks. A numerical evaluation of the scatter between the discrete model with different staggered tiers and a continuous model obtained from the homogenization procedure is carried out the limiting case is the stack bond and running bond textures.

KEY WORDS: *discrete element model, homogenization, periodic brickwork, in plane actions*

1. INTRODUCTION

Historical masonry constructions were built with different textures. Masonries may be considered regular when brick or stone blocks are regularly shaped and characterized by disposition of units along horizontal lines. Masonries may be considered irregular when they are composed of stones irregular in shape and disposition. While for the case of a periodic regular texture there are many papers (Alpa and Monetto, 1994; Anthoine and Pegon, 1994; Lee et al., 1996, 1998; Luciano and Sacco, 1997; Masiani et al., 1995; Lourenço and Rots, 1993; Cecchi and Sab, 2002; Cecchi, 2010), it is worth noting that, as far as masonry with an irregular texture is concerned, there is still a lack of literature, a reference can be found in (Falsone and Lombardo, 2007; Cluni and Gusella, 2004).

In particular, the analysis carried out by Cluni and Gusella (2004) refers to irregular textures that are treated at a micro-mechanical scale by means of the Rep-

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resentative Elementary Volume (REV) – finite size test-windows with variable dimensions. The test-windows are placed inside a masonry in different and arbitrary positions. For each position the macroscopic masonry stiffness is evaluated. Then the ensemble average of macromechanical masonry stiffness is estimated. The procedure is brought to a stop when the dimensions of the test-windows are large enough to minimize the variance between values of the elastic stiffness evaluated. In the present paper, starting from a periodic running bond pattern, and a random model proposed by Cecchi and Sab (2009), a perturbation only in the position of vertical interfaces is introduced, while the block dimensions are fixed. In this way, masonry with different staggered tiers is obtained.

Two models have been constructed: a discrete one and a continuous one. The discrete model is based on the implementation of a numerical model already formulated in the case of a regular periodic running bond masonry (Cecchi and Sab, 2004) and a random masonry (Cecchi and Sab, 2009). The blocks which form the masonry wall are modeled as rigid bodies connected by elastic interfaces (mortar thin joints). Different textures are obtained as a function of the α parameter that represents different translations among vertical interfaces.

The continuous model is based on the homogenization procedure. Explicit equations for constitutive functions have been obtained. The limiting case in masonry textures are running and stack bond patterns. A comparison between the continuous and discrete models is crucial for evaluating the role of the bond in the structural masonry behavior.

2. THE DISCRETE MODEL

Consider a standard running bond periodic masonry with the block dimensions a (height), b (width), and t (thickness). The idea is to superimpose a perturbation on the y_1 horizontal positions of the vertical interfaces by moving them at random as described in Fig. 1b. Hence, in the random model, all blocks have the same height a and the same thickness t, while the width b is random (Fig. 1a,b).

The general procedure was described by Cecchi and Sab (2009). Here, the idea is to hold the block width b constant, and move the position of vertical interfaces (Fig. 1d). In this way, the limiting solutions represent the running bond and stack bond pattern. The variable α defined as $\alpha = pb$ is introduced, where p = 0–0.5 is a parameter. When p is equal to zero, the running bond solution is obtained; when $p = 0.5$ the stack bond solution is obtained. The latter case is the limiting condition in which each block has only four neighbouring blocks. In Fig. 2, the stagger α is shown.

Let $\mathbf{y}^{i,j}$ be the position of the generic block $B_{i,j}$ (Fig. 1a) in the Euclidean space. As shown by Fig. 1a, j can actually take arbitrary values while i is such as $i + j$ is even. The displacement of each block is the motion of a rigid body:

$$\mathbf{u} = \mathbf{u}^{i,j} + \mathbf{\Omega}^{i,j} \times (\mathbf{y} - \mathbf{y}_p^{i,j}) \, , \, \forall \, \mathbf{y} \in B_{i,j} \, , \tag{1}$$

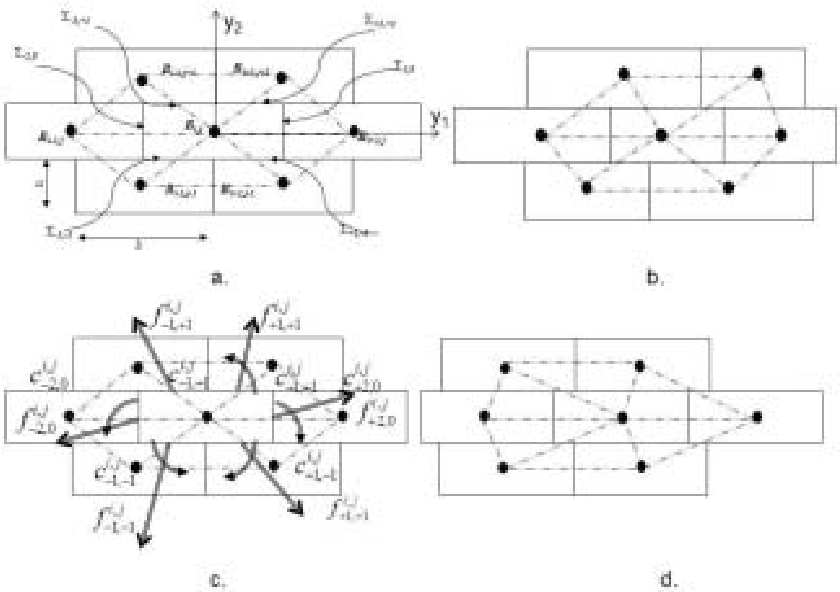

FIG. 1: Discrete model

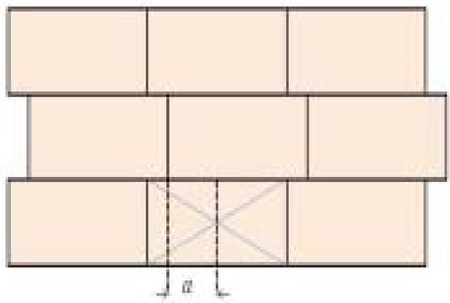

FIG. 2: Staggered tiers

$\mathbf{u}^{i,j}$ is the translation vector and $\boldsymbol{\Omega}^{i,j}$ is the rotation vector of $B_{i,j}$.

Due to the regularity of the masonry structure, the $B_{i,j}$ block interacts with the $B_{i+k_1,j+k_2}$ block by means of Σ_{k_1,k_2} elastic joints as follows:

- if $k_1, k_2 = \pm 1$, then Σ_{k_1,k_2} is a horizontal interface;
- if $k_1 = \pm 2$ and $k_2 = 0$, then Σ_{k_1,k_2} is a vertical interface. The interfaces of the $B_{0,0}$ block are

$$\Sigma_{-1,-1}=\begin{pmatrix} -\frac{b}{2}\le y_1\le \alpha \\ y_2=-\frac{a}{2} \\ -\frac{t}{2}\le y_3\le +\frac{t}{2}\end{pmatrix};\quad \Sigma_{+1,-1}=\begin{pmatrix} \alpha\le y_1\le \frac{b}{2} \\ y_2=-\frac{a}{2} \\ -\frac{t}{2}\le y_3\le +\frac{t}{2}\end{pmatrix}$$

$$\Sigma_{-1,+1}=\begin{pmatrix} -\frac{b}{2}\le y_1\le \alpha \\ y_2=\frac{a}{2} \\ -\frac{t}{2}\le y_3\le +\frac{t}{2}\end{pmatrix};\quad \Sigma_{+1,+1}=\begin{pmatrix} \alpha\le y_1\le \frac{b}{2} \\ y_2=\frac{a}{2} \\ -\frac{t}{2}\le y_3\le +\frac{t}{2}\end{pmatrix}$$

$$\Sigma_{-2,0}=\begin{pmatrix} y_1=-\frac{b}{2} \\ -\frac{a}{2}\le y_2\le +\frac{a}{2} \\ -\frac{t}{2}\le y_3\le +\frac{t}{2}\end{pmatrix};\quad \Sigma_{-2,0}=\begin{pmatrix} y_1=\frac{b}{2} \\ -\frac{a}{2}\le y_2\le +\frac{a}{2} \\ -\frac{t}{2}\le y_3\le +\frac{t}{2}\end{pmatrix}$$

where b is the width of the block and a is its height. If the mortar joint is modeled as an elastic interface – such a problem has been considered by Klarbring (1991) by means of perturbative techniques – the deformation between two blocks may be written as a function of the $[[\mathbf{u}]]$ displacement jump. The constitutive prescription for the contact is a linear relation between the tractions on the block surfaces and the jump of the displacement field.

$$\sigma\mathbf{n}=K[[\mathbf{u}]]\ ;\quad \text{on } \Sigma_{k_1,k_2}\ . \tag{2}$$

Here σ is the stress tensor, $\mathbf{n}$ is the normal to Σ_{k_1,k_2}, $[[\mathbf{u}]]$ is the jump of the displacement field at Σ_{k_1,k_2}, and $\mathbf{K}$ is given by

$$\mathbf{K}_{ij}=\frac{1}{e}\,\mathbf{a}^{M}_{iklj}\mathbf{n}_k\mathbf{n}_l\ , \tag{3}$$

where $\mathbf{a}^{\mathbf{M}}$ is the elastic stiffness tensor of the mortar and e is the thickness of the real joint. It will be assumed in what follows that the mortar is isotropic, then Eq. (3) becomes

$$\begin{aligned}
\mathbf{K}_h(t)&=\frac{1}{e^h}\left(\frac{E_h^{\ m}}{2\left(1+\nu_h^{\ m}\right)}\mathbf{I}+\left(\frac{E_h^{\ m}}{2\left(1+\nu_h^{\ m}\right)}+\frac{\nu_h^{\ m}E_h^{\ m}}{\left(1+\nu_h^{\ m}\right)\left(1-2\nu_h^{\ m}\right)}\right)(\mathbf{n}\otimes\mathbf{n})\right),\\
\mathbf{K}_v(t)&=\frac{1}{e^v}\left(\frac{E_v^{\ m}}{2\left(1+\nu_v^{\ m}\right)}\mathbf{I}+\left(\frac{E_v^{\ m}}{2\left(1+\nu_v^{\ m}\right)}+\frac{\nu_v^{\ m}E_v^{\ m}}{\left(1+\nu_v^{\ m}\right)\left(1-2\nu_v^{\ m}\right)}\right)(\mathbf{n}\otimes\mathbf{n})\right).
\end{aligned} \tag{4}$$

Here e^h and e^v are the thickness of the actual horizontal and vertical joints and **n** is the normal to the interface; E_h^m and E_v^m are the Young moduli and ν_h^m and ν_v^m are the Poisson ratios of the mortar (Klarbring, 1991). In this way different constitutive functions may be assumed for horizontal and vertical joints. This assumption can be used for masonry in actual building technique to justify different stiffnesses in horizontal and vertical bed joints. Note that the **K** tensor has a diagonal form in this case.

If the in-plane case is considered, the vector of the degrees of freedom is $\mathbf{u} = (u_1^{i,j}, u_2^{i,j}, \Omega_3^{i,j})^T$. The interactions between the blocks through the interfaces are represented by the elastic forces $\mathbf{f}^{i,j}$ and the $c^{i,j}$ moment, $\mathbf{f} = (f_1^{i,j}, f_2^{i,j}, c^{i,j})^T$, which is to be found. The following notations are introduced:

$$\Delta_1^{k_1,k_2} = u_1^{i+k_1,j+k_2} - u_1^{i,j} + k_2 a \frac{\Omega_3^{i+k_1,j+k_2} + \Omega_3^{i,j}}{2}, \tag{5}$$

$$\Delta_2^{k_1,k_2} = u_2^{i+k_1,j+k_2} - u_2^{i,j} - \left(k_1 \frac{b}{2} + \alpha\right) \frac{\Omega_3^{i+k_1,j+k_2} - \Omega_3^{i,j}}{2}, \tag{6}$$

$$\delta_3^{k_1,k_2} = \Omega_3^{i+k_1,j+k_2} - \Omega_3^{i,j}. \tag{7}$$

2.1 Horizontal Interfaces ($k_1 = \pm 1$, $k_2 = \pm 1$):

Let e^h be the thickness of the real horizontal joint, $S_h^{k_1,k_2} = \left| k_1 \frac{b}{2} - \alpha \right| t$ be the area of the horizontal interface, and $I_{h3}^{k_1,k_2} = \frac{\left| k_1 \frac{b}{2} - \alpha \right|^3 t}{12}$ be its inertia with respect to the y_3 axis (that is orthogonal to the middle 2D plane of masonry).

Using [[**u**]] to denote the jump of the displacement field at the Σ_{k_1,k_2} interface, the following expression of the horizontal interface elastic energy can be obtained:

$$W^{k_1,k_2} = \frac{1}{2} \int_{\Sigma_{k_1,k_2}} [[\mathbf{u}]]\mathbf{K}[[\mathbf{u}]] = \frac{K_h''}{2e^h} \left(S_h^{k_1,k_2} \left[(\Delta_1^{k_1,k_2})^2 \right] \right)$$
$$+ \frac{K_h'}{2e^h} \left(\left(S_h^{k_1,k_2} (\Delta_2^{k_1,k_2})^2 + I_{h3}^{k_1,k_2} (\delta_3^{k_1,k_2})^2 \right] \right), \tag{8}$$

where $K' = E^m(1 + \nu^m)\ (1 - 2\nu^m)$ and $K'' = E^m/(2(1 + \nu^m))$. The quantity h identifies horizontal interfaces. The forces and moment that the $B_{i+k_1,j+k_2}$ block applies to the $B_{i,j}$ block (Fig. 1c) are

$$\left(f_1\right)_{i,j}^{k_1,k_2} = -\frac{\partial W^{k_1,k_2}}{\partial u_1^{i,j}} = \frac{K_h''}{e^h} S_h^{k_1,k_2} \Delta_1^{k_1,k_2}, \tag{9}$$

$$\left(f_2\right)_{i,j}^{k_1,k_2} = -\frac{\partial W^{k_1,k_2}}{\partial u_2^{i,j}} = \frac{K_h'}{e^h} S_h^{k_1,k_2} \Delta_2^{k_1,k_2}, \tag{10}$$

$$\left(c\right)_{i,j}^{k_1,k_2} = -\frac{\partial W^{k_1,k_2}}{\partial \Omega_3^{i,j}} = \frac{K_h'}{e^h} I_{h3}^{k_1,k_2} \delta_3^{k_1,k_2}$$

$$-\frac{K_h''}{2e^h} S_h^{k_1,k_2} k_2 a \Delta_1^{k_1,k_2} + \frac{K_h'}{2e^h} S_h^{k_1,k_2} \left(k_1 \frac{b}{2} + \alpha\right) \Delta_2^{k_1,k_2}. \tag{11}$$

2.2 Vertical Interfaces ($k_1 = \pm 2$, $k_2 = 0$)

Let e^v be the thickness of the real vertical joint, $S_v = at$ be the area of the vertical interface, and $I_{v3} = a^3 t/12$ be its inertia with respect to the y_3 axis. The following expression of the vertical interface elastic energy can be obtained:

$$W^{k_1,k_2} = \frac{1}{2} \int_{\Sigma_{k_1,k_2}} [[\mathbf{u}]]\mathbf{K}[[\mathbf{u}]] = \frac{K_v''}{2e^v} S_v \left[(\Delta_2^{k_1,k_2})^2 \right]$$

$$+ \frac{K_v'}{2e^v} \left(\left(S_v (\Delta_1^{k_1,k_2})^2 + I_{v3} (\delta_3^{k_1,k_2})^2 \right) \right). \tag{12}$$

The quantity v identifies horizontal interfaces. For the in-plane case, the forces and moment that the $B_{i+k_1,j+k_2}$ block applies to the $B_{i,j}$ block (Fig. 1c) are

$$\left(f_1\right)_{i,j}^{k_1,k_2} = -\frac{\partial W^{k_1,k_2}}{\partial u_1^{i,j}} = \frac{K_v'}{e^v} S_v \Delta_1^{k_1,k_2}, \tag{13}$$

$$\left(f_2\right)_{i,j}^{k_1,k_2} = -\frac{\partial W^{k_1,k_2}}{\partial u_2^{i,j}} = \frac{K''_v}{e^v} S_v \Delta_2^{k_1,k_2}, \tag{14}$$

$$\left(c\right)_{i,j}^{k_1,k_2} = -\frac{\partial W^{k_1,k_2}}{\partial \Omega_3^{i,j}} = \frac{K'_v}{e^v} I_{v3}^{k_1,k_2} \delta_3^{k_1,k_2} + k_1 \frac{K'_v}{4e^v} S_v^{k_1,k_2} b \Delta_2^{k_1,k_2}. \tag{15}$$

It must be noted that according to the chosen perturbation, the vertical interfaces are defined in a deterministic manner.

At the present time, one can easily check that the in-plane elastic actions $F_{elastic}$ can be written as

$$\mathbf{F}_{elastic} = -\partial W / \partial \mathbf{u} = -\mathbf{K}\,\mathbf{u} = -\mathbf{F}_{ext}. \tag{16}$$

Here W is the total elastic energy, $\mathbf{F}_{ext}$ is the vector of the applied in-plane actions, $\mathbf{K}$ is the in-plane stiffness matrix. Further details are given in Appendix.

3. HOMOGENIZED SOLUTION

The continuous model can be obtained starting from the equations of potential energy (8), (12) with periodicity boundary conditions, under a hypothesis of a rigid block. In other words, $\boldsymbol{\varepsilon} = 0$ means that the corresponding displacement is piecewise rigid on each block with a possible jump at the interface. The displacement field corresponding to ε is defined by

$$ {}^{s}\nabla\mathbf{u} = \mathbf{E} + {}^{s}\nabla\mathbf{u}^{\mathbf{per}}, \tag{17}$$

where $\mathbf{E} = \mathbf{E}_{\alpha\beta}$ is the membrane in-plane strain tensor. Therefore, until the rigid body sets in motion, the displacement field is

$$\mathbf{u} = \begin{pmatrix} E_{11}y_1 + E_{12}y_2 \\ E_{12}y_1 + E_{22}y_2 \end{pmatrix}. \tag{18}$$

Since $\boldsymbol{\varepsilon} = 0$ and $\mathbf{u}$ are discontinuous at the interface, then $\mathbf{u}$ is rigid on each block:

$$\mathbf{u}(\mathbf{y}) = \mathbf{t}^i + \boldsymbol{\Omega}^i \times (\mathbf{y} - \mathbf{y}^{i,j}) \;\; \forall \;\; \mathbf{y} \in B_{i,j}, \tag{19}$$

where $\mathbf{y}^{i,j}$ is the center of the ith block, t^i is its translation; [[$\mathbf{u}$]] is the jump of the displacement field which is assumed to be the deformation measure. The principle of stationarity gives the value of the unknown rotation Ω_3:

$$\min_{\Omega_3} W = \frac{1}{2}\mathbf{E}\cdot(\mathbf{A}^{\mathbf{F}}\mathbf{E}). \tag{20}$$

By minimizing it, the Ω_3 unknown is

$$\frac{\partial W}{\partial \Omega_3} = 0 \Rightarrow \Omega_3 = \frac{K_v^{''}\frac{e^h}{a} - K_h^{''}\frac{e^v}{b} + K_h^{'}\frac{e^v}{a}\left(\frac{b}{4a} - \frac{\alpha^2}{ab}\right)}{K_v^{''}\frac{e^h}{a} + K_h^{''}\frac{e^v}{b} + K_h^{'}\frac{e^v}{a}\left(\frac{b}{4a} - \frac{\alpha^2}{ab}\right)} E_{12}. \tag{21}$$

If $\alpha = 0$, the running bond value of Ω_3 is obtained:

$$\frac{\partial W}{\partial \Omega_3} = 0 \Rightarrow \Omega_3 = \frac{K_v^{''}\frac{e^h}{a} - K_h^{''}\frac{e^v}{b} + \frac{b}{4a}K_h^{'}\frac{e^v}{a}}{K_v^{''}\frac{e^h}{a} + K_h^{''}\frac{e^v}{b} + \frac{b}{4a}K_h^{'}\frac{e^h}{a}} E_{12}. \tag{22a}$$

If $\alpha = b/2$, the stack bond value of Ω_3 is obtained:

$$\frac{\partial W}{\partial \Omega_3} = 0 \Rightarrow \Omega_3 = \frac{K_v^{"} \dfrac{e^h}{a} - K_h^{"} \dfrac{e^v}{b}}{K_v^{"} \dfrac{e^h}{a} + K_h^{"} \dfrac{e^v}{b}} E_{12}\ . \tag{22b}$$

Hence the elastic homogenized membranal constants are

$$A_{1111}^F(\alpha) = \frac{4K_v^{'} \dfrac{e^h}{a} + K_h^{"} \dfrac{e^v}{a}\left(\dfrac{b}{a} - \dfrac{4\alpha^2}{ab}\right)}{4\dfrac{e^v}{b}\dfrac{e^h}{a}}, \tag{23}$$

$$A_{1122}^F(\alpha) = 0, \tag{24}$$

$$A_{2222}^F(\alpha) = \frac{K_h^{'}}{\dfrac{e^h}{a}}, \tag{25}$$

$$A_{1212}^F(\alpha) = K_h^{"} \frac{K_h^{'} \dfrac{e^v}{b}\left(1 - \dfrac{4\alpha^2}{b^2}\right) + \dfrac{4a}{b} K_h^{"} \dfrac{e^h}{b}}{\dfrac{e^h}{a}\left(K_h^{'} \dfrac{e^v}{b}\left(1 - \dfrac{4\alpha^2}{b^2}\right) + \dfrac{4a}{b} K_h^{"} \dfrac{e^h}{b} + \dfrac{4a^2}{b^2} K_h^{"} \dfrac{e^v}{b}\right)}. \tag{26}$$

Equations (23)–(26) clearly show that, as expected, the quantity A_{2222}^F does not depend on α, hence this coefficient is the same for stack and running bond textures. While A_{1111}^F and A_{1212}^F are functions of α, in particular, if $\alpha = 0$, the running bond masonry coefficients are obtained:

$$A_{1111}^F(\alpha) = \frac{4K_v^{'} \dfrac{e^h}{a} + \dfrac{b}{a} K_h^{"} \dfrac{e^v}{a}}{4\dfrac{e^v}{b}\dfrac{e^h}{a}}, \tag{27}$$

$$A_{1212}^F(\alpha) = K_h^{"} \frac{K_h^{'} \dfrac{e^v}{b} + \dfrac{4a}{b} K_h^{"} \dfrac{e^h}{b}}{\dfrac{e^h}{a}\left(K_h^{'} \dfrac{e^v}{b} + \dfrac{4a}{b} K_h^{"} \dfrac{e^h}{b} + \dfrac{4a^2}{b^2} K_h^{"} \dfrac{e^v}{b}\right)}, \tag{28}$$

if $\alpha = b/2$, the stack bond masonry coefficients are obtained:

$$A_{1111}^F(\alpha) = \frac{K_v^{'}}{\dfrac{e^v}{b}}, \tag{27}$$

$$A_{1212}^F(\alpha) = \frac{K_h^{"}}{\dfrac{e^h}{a} + \dfrac{e^v}{b}}. \tag{28}$$

4. COMPARISON BETWEEN DISCRETE AND HOMOGENIZED MODELS

To validate discrete model, a comparison with the homogenized model of the previous section has been made for some classical cases for which an analytical solution exists.

- CASE 1

A panel, simply supported at its left edge, is loaded with a horizontal uniform force $F_1 > 0$ which is applied to each block center (Fig. 3a). Using Eq. (23), a continuous homogenized solution is obtained.

The analytical solution gives a maximum displacement at the right edge of the panel:

$$u_1(x_1 = L) = \frac{F_1}{2A^F_{1111}(\alpha)abt} L^2 . \tag{29}$$

The relative error: $(u^{discrete}_{max} - u^{analytical}_{max})/u^{alytical}_{max}$ between this analytical solution and the average discrete solution is less than 5% for all the tested values of the parameter α and for the two cases of 121 heterogeneities and 225 heterogeneities.

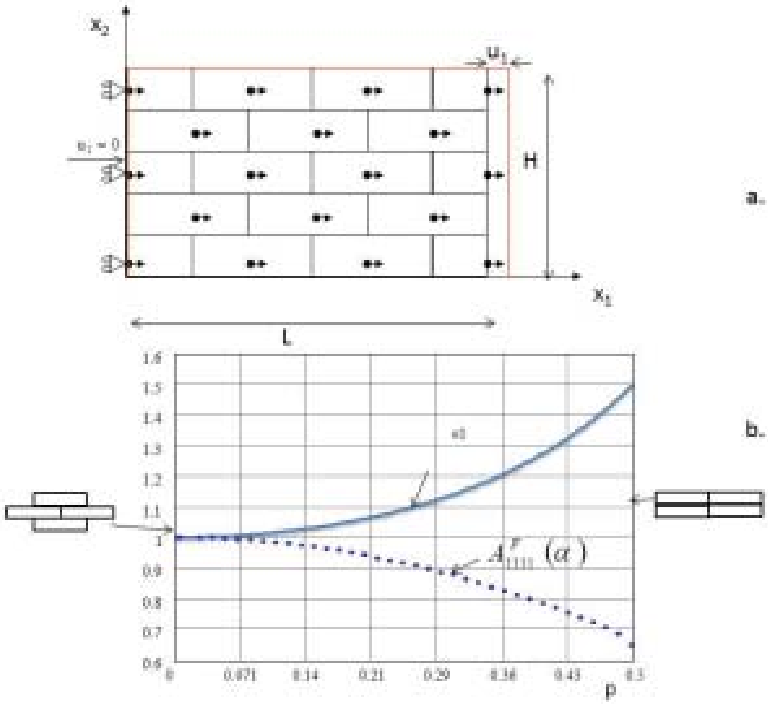

FIG. 3: Masonry panel (width L and height H) simply supported at the left edge and subjected to a horizontal uniform load

In Fig. 3b, the sensitivity to the texture is shown. In particular, in the ordinate A^F_{1111} the normalized versus the A^F_{1111} running bond and $u_1(x_1 = L)$ normalized versus its value in the case of the running bond are reported versus the parameter p. It must be noted that A^F_{1111} decreases by 37% from the running to the stack bond and $u_1(x_1 = L)$ displacement increases by 50% from the running to the stack bond. These results are in agreement with those obtained in the discrete model.

- CASE 2

The case of macroscopic uniform shear stress is investigated. Two cases are taken into account. One is the case of a panel under the following boundary conditions: blocks at the top of the panel: uniform force $F_{1ext} = F_1 > 0$, $F_{2ext} = 0$, $M_{ext} = 0$; blocks on the left side of the panel $u_2 = 0$ and u_1, ω_3 are free; blocks on the right side of the panel $u_2 = 0$ and u_1, ω_3 are free; blocks at the base of the panel under the condition $u_2 = u_1 = \omega_3 = 0$ (Fig. 4). Using Eq. (26), a continuous homogenized solution is obtained. For the two cases the analytical solution respectively is

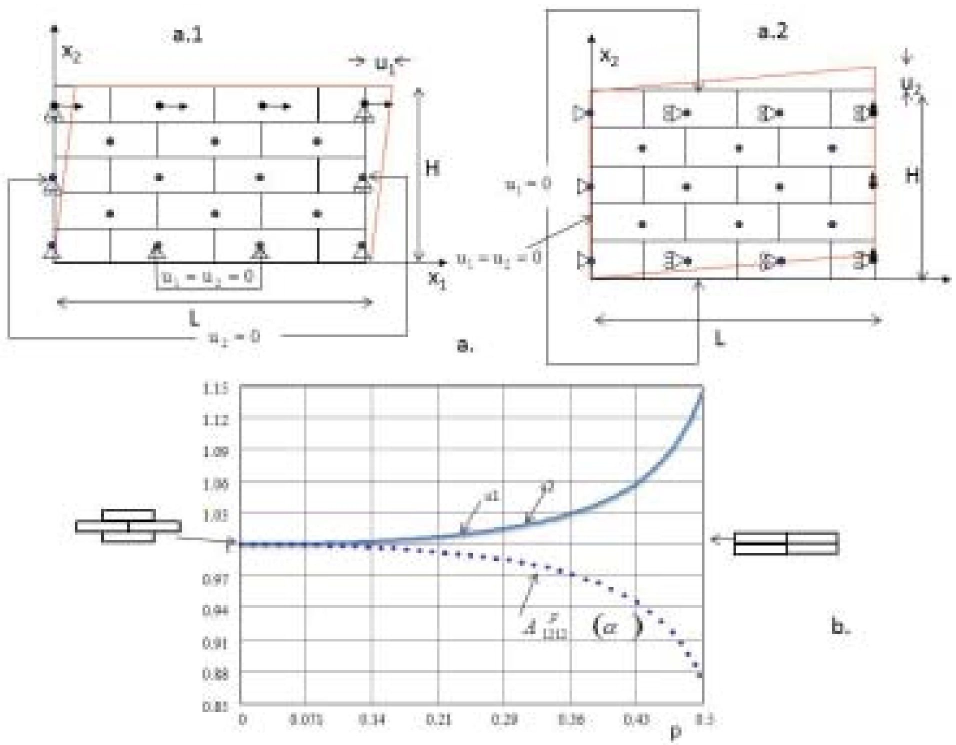

FIG. 4: Masonry panel (width L and height H) subjected to shear actions: a.1) simply supported at its left and right edges $u_2 = 0$, and hinged at the base loaded with a horizontal uniform force $F_1 > 0$ applied to the top, a.2) subject to shear actions hinged at the left edge, and simply supported $u_1 = 0$ at the base ant at the top, loaded with a vertical uniform force $F_2 > 0$ applied to the right side

$$u_1(x_1 = L, x_2 = H) = \frac{F_1 LH}{A^F_{1212}(\alpha)abt}\ ;\ u_2(x_1 = L, x_2 = H) = \frac{F_2 LH}{A^F_{1212}(\alpha)abt}\ . \tag{30}$$

The relative error, evaluated as in the previous case, between this analytical solution and the average discrete solution is less than 5% for all the tested values of the parameter p and for two cases of 121 and 225 heterogeneities as in previous case 1. According to the previous case, in Fig. 4b the sensitivity to texture is shown. In particular, in the ordinate A^F_{1212} normalized versus the A^F_{1212} running bond and $u_1(x_1 = L)$ and/or $u_1(x_2 = H)$ normalized versus its value in the case of the running bond are reported versus the parameter p. It must be noted that A^F_{1212} decreases by 23% from the running to the stack bond and displacement increases in the discrete model.

5. NUMERICAL EXAMPLE

At this point, technical cases are studied with the discrete model, and the capacity of this model to take into account the sensitivity to different textures is proposed.

- CASE 1

A panel hinged at the base and subjected to the F_1 force at the top is considered.

In this case, the discrete solution is studied for α which varies from 0 (the running bond) to $b/2$ (the stack bond). The width L = 1500 mm and height H = 1250 mm; in this case, the number of heterogeneities is 150 for the running bond case. In Fig. 5, the trend of horizontal displacement u_1 for $0 \le y_2 \le H$ is reported. The

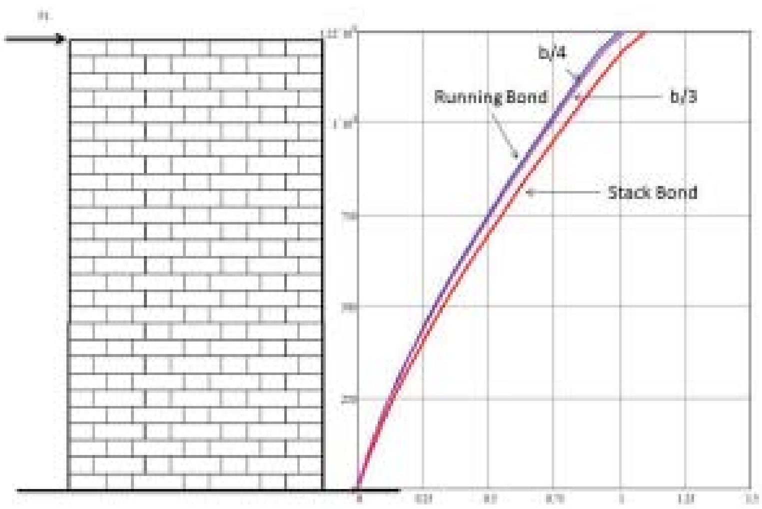

FIG. 5: Masonry panel (width L and height H) subjected to horizontal actions: trend of $u_1(\alpha)$

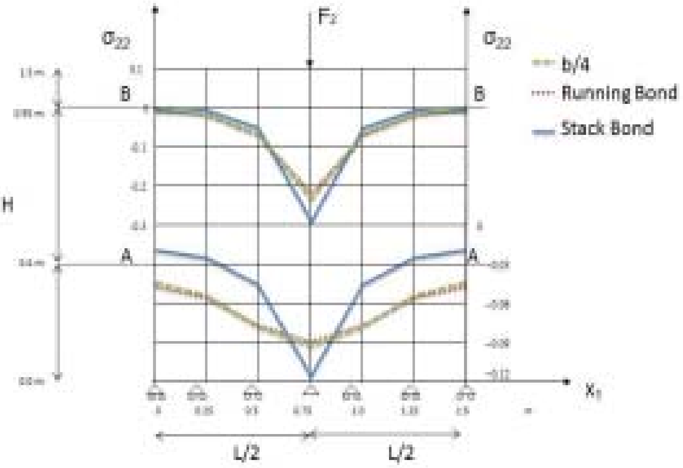

FIG. 6: Masonry panel (width L and height H in meters) subjected to the vertical force: trend of the $\sigma_{22}(\alpha)$ stress

relative error between the running bond and the stack bond cases is in the u_1 displacement at the panel bottom by 9.9%, for $\alpha = b/4$ the relative error between the running bond solution is equal to 1.1% and for $\alpha = b/3$ the difference is 2.16%.

- CASE 2

A simple panel supported at the base, hinged at the base center (such as to prevent rigid motions) and subjected to the F_2 force at the top is considered.

Also, in this case, the discrete solution is studied for α which varies from 0 (the running bond) to $b/2$ (the stack bond). The width L = 1500 mm and height H = 1500 mm; in this case, the number of heterogeneities is 180 for the running bond case. In Fig. 6, the trend of the vertical stress σ_{22} in two horizontal sections A–A for H = 400 mm and B–B for H = 950 mm is reported. The staggered effect in stress diffusion must be noted.

6. CONCLUSIONS

In this work, a comparison between the proposed numerical discrete model and a homogenized continuum model has been made. The homogenized model is based on an analytical approach. The results (both by means of a discrete model and the corresponding homogenized model) show that the running bond pattern always represents the upper bound for the stiffness of masonry. For this reason, such a masonry is called *a regola d'arte*. Moreover, the presence of the stagger allows stress diffusion, in this case, the masonry behavior may be assumed as a continuous material.

APPENDIX. Numerical Procedure

The same procedure as that proposed by Cecchi and Sab (2004) is here repurposed for the in-plane case and implemented to take into account the randomness of masonry. Although standard methods exist to solve numerically (16), the Molecular Dynamics method (Allen and Tildsey, 1994; Owen and Hinton, 1980) has been developed in the perspective of linear and non-linear analyses with dynamic loading. In this case, the equations to be solved are

$$\mathbf{u} = (u_\alpha^{i,j}, \Omega_3^{i,j})^T , \ \alpha = 1,3 , \tag{31}$$

$$\mathbf{M}\frac{\partial^2\mathbf{u}}{\partial t^2} + \mathbf{K}\left(\mathbf{uM} + \mu\frac{\partial\mathbf{u}}{\partial t}\right) = \mathbf{F}_{ext} , \tag{32}$$

where μ is the damping coefficient, $\mathbf{F}_{ext}$ are the applied actions, $\mathbf{M}$ is the (diagonal) mass matrix, and $\mathbf{K}$ the stiffness matrix.

To solve the dynamic equation (32), the predictor–corrector algorithm GEAR of order 2 is used. Let $\mathbf{u}(t)$, $\mathbf{v}(t)$ and $\mathbf{A}(t)$ denote the displacement, the velocity and the acceleration at time t. Using a Taylor expansion, the correspondent predictor vectors at time $t + \delta t$ are

$$\mathbf{u}^p(t + \delta t) = \mathbf{u}(t) \_ \delta t{\cdot}\mathbf{v}(t)\frac{1}{2}\delta t^2{\cdot}\mathbf{A}(t) + o(\delta t^3) , \tag{33}$$

$$\mathbf{v}^p(t + \delta t) = \mathbf{v}(t) + \delta t{\cdot}\mathbf{A}(t) + o(\delta t^2) , \tag{34}$$

$$\mathbf{A}^p(t + \delta t) = \mathbf{A}(t) + o(\delta t^2) , \tag{35}$$

Using the balance equation, the real accelerations may be found:

$$\mathbf{A} = \mathbf{M}^{-1}(\mathbf{f}_{ext} - \mathbf{K}(\mathbf{u}^p + \mu\mathbf{v}^p)) , \tag{36}$$

and the error at the predictor time step may be calculated:

$$\delta\mathbf{A}(t + \delta t) = \mathbf{A} - \mathbf{A}^p(t + \delta t) . \tag{37}$$

Finally, the corrector time step is introduced:

$$\mathbf{u}(t + \delta t) = \mathbf{u}^p(t + \delta t) + \frac{1}{4}4\delta t^2{\cdot}\delta\mathbf{A}{\cdot}c_0 , \tag{38}$$

$$\mathbf{v}(t + \delta t) = \mathbf{v}^p(t + \delta t) + \frac{1}{2}\delta t{\cdot}\delta\mathbf{A}{\cdot}c_1 , \tag{39}$$

$$\mathbf{A}(t + \delta t) = \mathbf{A}^p(t + \delta t) + \delta\mathbf{A}{\cdot}c_2 , \tag{40}$$

where $c_0 = 0$, $c_1 = 1$ and $c_2 = 1$. In the special case of static equilibrium, the time step integration is stopped when

$$e_{num} = \|\mathbf{K}\mathbf{u}^p - \mathbf{F}_{ext}\| < toler \,. \tag{41}$$

In the above formula, the force balance equations are normalized by a typical applied force (for instance, ρab and the moment balance equations are normalized by a typical applied moment (for instance, ρab^2). In this case, the norm used is the maximum absolute value and the tolerance is 0.005. In other words, the maximum error in the force balance equations is less than 0.005 ρab and the maximum error in the moment balance equations is less than 0.005 ρab^2.

The time step δt must be much smaller than the critical value T_c calculated as a function of the mass and stiffness properties of the block. So

$$\delta t = T_c 100 \,, \; T_c = \sqrt{mk_n} \,, \; k_n = S_h K' e^v \,, \; m = \rho abt \,, \; \mu = \sqrt{mk_n} \,, \tag{42}$$

where ρ is the density of the block.

The software developed here is written in Fortran. The program formulation starts from geometrical description of a generic masonry wall. Each block is identified with its center position. As shown in Fig. 7 a wall with k-block courses has been investigated: odd courses present n-blocks, while even courses present $n + 1$ blocks — hence, the length between two even courses and two odd courses is $2n + 1$.

The following steps are proposed:

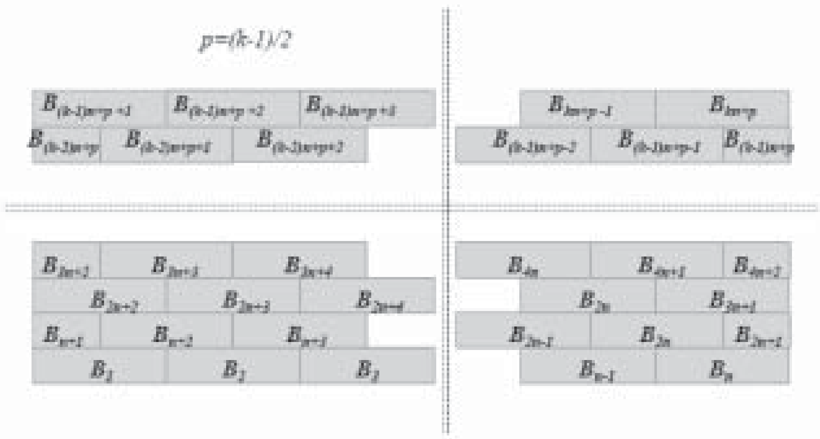

FIG. 7: Geometrical description of a generic masonry wall

- definition of geometrical and mechanical quantities: $y^{i,j}$ for different values of p;
- construction of mass tensor for the generic i-block;
- imposition of the boundary condition on forces — applied loads — and on displacement — constraint degrees of freedom;
- step 0: the initial displacements, velocities and accelerations are set to zero;
- step i: computation of the predicted displacements, velocities and accelerations (124–126);
- evaluation of elastic and damping forces and moments at the interfaces according to Section 2 procedure;
- if the static equilibrium according to Eq. (41) is satisfied, then we pass on to the evaluation of $\varepsilon_{\mathbf{N}}$; if $\varepsilon_{\mathbf{N}} \leq 0.03$ we stop;
- evaluation of real acceleration at the i-step according to Eq. (36);
- evaluation of the corrected displacements, velocities and accelerations, Eqs. (38)–(40);
- pass on to step $\mathbf{i} + 1$.

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