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An effective Galerkin Boundary Element Method for a 3D half-space subjected to surface loads

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Abstract: - In this work, a simple and effective numerical model is proposed for solving the problem of a three-dimensional elastic and isotropic half-space subjected to surface vertical displacements and pressures. For this purpose, the Galerkin Boundary Element Method for a three-dimensional half-space is introduced, and both surface displacement and pressure fields are discretized in order to obtain fast and accurate numerical solution. Assuming a piecewise constant discretization of both surface displacement and pressure fields, several numerical tests are performed showing the effectiveness of the model, for instance by determining accurately the translational and rotational stiffness of a rigid rectangular foundation on elastic half-space, together with the displacements generated by a uniform surface pressure over a rectangular area.

Key-Words: - Elastic half-space, Boundary Element Method, Boussinesq solution, rigid foundation stiffness, rigid indenter, half-space surface displacement, half-space surface pressure.

1 Introduction

The three-dimensional (3D) elastic half-space is one of the most accurate models for representing the behavior of a semi-infinite elastic and isotropic continuum, which can be adopted, for instance, for describing the response of a soil media subjected to external loads. The use of a continuum model is accurate since it considers surface deflections occurring not only under the directly loaded regions, but also within certain areas outside the loaded regions, as the common experience can suggest [1].
In this field of analysis, the pioneering research of Cerruti [2] and Boussinesq [3] introduced the potential of an elastic and isotropic 3D half-space, that allowed to obtain the expressions of stresses and displacements generated by a concentrated force normal to half-space surface [4]. The problem of the analytic determination of the displacements generated by various force distributions on the surface of the half-space was studied by many researchers in the past [1]. Among the others, Love [5] determined the expression of half-space surface displacements generated by a uniform pressure over a rectangular area. The indentation of a rigid punch on the half-space represents another problem which involves Boussinesq solution. This problem is strictly related to the determination of the dynamic stiffness of a rigid rectangular foundation resting on an elastic soil, and it is also a classical problem in physics, since its solution represents the charge density of a thin electrified plate. Many researchers determined the solution of this problem by adopting different approaches such as power series or the Boundary Element Method [6], [7], [8], [9], [10], [11], [12]. A brief but complete resume of some numerical and analytical solutions of problems related to half-space surface loads and rigid loaded areas can be also found in the book by Poulos and Davis [13]. The analysis of 3D half-space behavior for soil-structure interaction purposes is still an active field of research [14].
In this work, the Galerkin Boundary Element Method is adopted for the determination of half-space surface displacements and/or pressures generated by different half-space surface loads/imposed displacements. For this purpose, the numerical model presented envisages a mixed formulation which assumes as independent fields both half-space surface normal pressures and vertical displacements. Such fields are both numerically approximated by means of a piecewise constant function in surface plane directions, and the loaded surface is subdivided for simplicity into rectangular portions, with particular attention to the use of power-graded discretization types instead of the regular ones. This numerical approach was recently adopted for discretizing the contact surface of an elastic beam on 3D half-space [15], obtaining fast accurate results. In particular, in [15] a standard three-dimensional Finite Element Model (FEM) was

also introduced for highlighting the effectiveness of the proposed mesh reduction method. Here, the numerical tests focus only on the determination of surface pressures generated by given simple surface displacements or the determination of surface displacements given by simple surface pressure distributions, and results are compared with existing analytical and numerical solutions.

Further developments of this work will focus on the development of plate models on 3D half-space, in order to simulate the behavior of plane shallow foundations on elastic soil.

2 Elastic half-space model

In this work, a 3D elastic, homogeneous, and isotropic half space, characterized by an elastic modulus E_s and a Poisson's ratio ν_s is considered. The half-space (Fig.1) is referred to a Cartesian coordinate system (O,x,y,z), having axis z directed downward; then z=0 represents the surface of the half-space. Considering a generic area S on the half-space surface, it can be subjected to various kinds of surface pressure distributions; however in the following, only a (vertical) normal pressure $p(x,y,0)=p_0(x,y)$ is considered, and only the corresponding vertical surface displacements $v(x,y,0)=v_0(x,y)$ are taken in consideration.

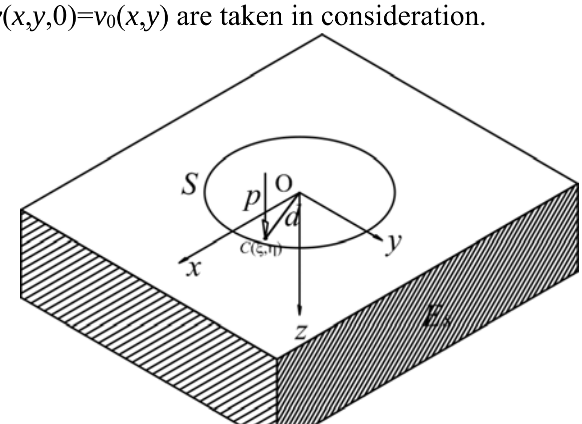


Fig.1 Half-space subjected to a concentrated vertical force p on its surface

The classic approach for the determination of stresses and displacements in an elastic 3D half-space due to surface forces was studied by Boussinesq [3] and Cerruti [2] by adopting the theory of potential. The following expression defines the vertical surface displacement $v_0(x,y)$, generated by the action of a normal pressure $p_0(\xi,\eta)$ over the generic area S:

$$v_0(x,y) = \int_S g(x,y,\xi,\eta)\; p_0(\xi,\eta)\, \mathrm{d}\xi \mathrm{d}\eta\,, \tag{1}$$

where $g(x,y,\xi,\eta)$ is the solution in terms of vertical surface displacement generated by a unitary normal force applied in a generic point $C(\xi,\eta)$ on half-space surface [4]:

$$g(x,y,\xi,\eta) = \frac{1}{\pi E_0} \frac{1}{[(x-\xi)^2 + (y-\eta)^2]^{1/2}}\,, \tag{2}$$

where $E_0 = E_s/(1-\nu_s^2)$. In the above equation, the expression $[(x\text{-}\xi)^2+(y\text{-}\eta)^2]^{1/2}$ represents the distance $d(x,y,\xi,\eta)$ of point C from the origin of the coordinate system. It is worth noting that $g(x,y,\xi,\eta)$ represents the deformed half-space surface given by a unitary force, which is a hyperboloid tending to an infinite vertical displacement under the point C and is asymptotic to the undeformed surface far from point C.

3 A simple Galerkin Boundary Element Method for half-space surface displacements and pressures

Simple problems such as the determination of half-space surface displacements due to constant pressures applied to simple or regular areas (for instance, rectangular), or, vice-versa, the determination of the surface pressures generated by known simple surface displacements given by rigid indenters, may be solved analytically starting from (1). However, many problems cannot be solved analytically, and numerical procedures can be adopted starting from (1). For instance, a Galerkin approach can be adopted for obtaining numerical solutions of (1), by introducing the following bilinear form and inner product:

$$B(p_0,\overline{p}_0) = B(\overline{p}_0,p_0) =$$

$$= \int_S \int_S g(x,y,\xi,\eta)\, p_0(\xi,\eta)\, \overline{p}_0(x,y)\; \mathrm{d}\xi \mathrm{d}\eta\, \mathrm{d}x \mathrm{d}y \tag{3}$$

$$(v_0,\overline{p}_0) = \int_S v_0(x,y)\, \overline{p}_0(x,y) \mathrm{d}x\mathrm{d}y \tag{4}$$

Then, the weak form of (1) can be written as follows:

$$(v_0,\overline{p}_0) = B(p_0,\overline{p}_0)\,. \tag{5}$$

In this work, a simple Galerkin discretization is adopted by subdividing the loaded area S of half-space surface into portions having a simple shape, namely a triangular or rectangular one. In particular in the following, rectangles with length h_{xi} and height h_{yi} are considered together with a piecewise constant base function over the generic i-th surface portion for discretizing surface displacements and pressures:

$$\rho_i(x,y)=\begin{cases}1 & \text{on the } i\text{-th element}\\ 0 & \text{elsewhere on } S\end{cases} \qquad (6)$$

Hence, the half-space surface pressure and the vertical displacement for each i-th element can be approximated as

$$p_0(x,y)=[\rho(x,y)]^{\mathrm{T}}\mathbf{r}_i, \qquad (7)$$

$$v_0(x,y)=[\rho(x,y)]^{\mathrm{T}}\mathbf{q}_i. \qquad (8)$$

where $\mathbf{r}_i$ and $\mathbf{q}_i$ are the vector components of surface pressures and vertical displacements. In particular, each component $\mathbf{r}_i$ represents a uniform surface pressure on the i-th surface portion, whereas each component $\mathbf{q}_i$ is a vertical displacement lumped at the centre of the corresponding i-th surface portion.

Substituting (7) and (8) into (5), the weak problem written in discrete form assumes the following expression:

$$\mathbf{Hq} = \mathbf{Gr}, \qquad (9)$$

which describes a mixed problem characterized by unknown surface displacement and pressure fields. Hence, surface vertical displacements $\mathbf{q}$ can be prescribed, and pressures $\mathbf{r}$ can be determined, or vice-versa. The components of matrices $\mathbf{H}$ and $\mathbf{G}$ are:

$$h_{ij}=\int_{y_i}^{y_{i+1}}\int_{x_i}^{x_{i+1}}\rho_i\rho_j\,\mathrm{d}x\mathrm{d}y=\begin{cases}(x_{i+1}-x_i)(y_{i+1}-y_i) & i=j\\ 0 & i\neq j\end{cases} \qquad (10)$$

$$g_{ij}=\frac{-1}{\pi E_0}\int_{y_i}^{y_{i+1}}\int_{x_i}^{x_{i+1}}\rho_i\,\mathrm{d}x\mathrm{d}y\int_{\eta_i}^{\eta_{i+1}}\int_{\xi_i}^{\xi_{i+1}}\frac{\rho_j}{d(x,y,\xi,\eta)}\mathrm{d}\xi\mathrm{d}\eta, \qquad (11)$$

where $(x_i,x_{i+1},y_i,y_{i+1})$ are the coordinates of the vertices of the i-th surface element and $(\xi_i,\xi_{i+1},\eta_i,\eta_{i+1})$ are the coordinates of the vertices of j-th surface element. It can be observed that matrix $\mathbf{H}$ turns out to be diagonal, having components along the diagonal equal to the area of each surface portion, whereas matrix $\mathbf{G}$ turns out to be more complex and fully populated. Details about its components are reported in [15] and [16].

3.1 Half-space surface discretization

In this work and in the following numerical tests, a rectangular area S is considered, L_1 and L_2 are area length and width in x and y direction, respectively, whereas the area is assumed to be centered with respect to the origin of the coordinate system:

$$S=\{(x,y,z): -L_1/2\leq x\leq L_1/2,\ -L_2/2\leq y\leq L_2/2,\ z=0\}. \qquad (12)$$

The surface S is subdivided into rectangular elements and the simplest discretization is obviously the regular one with equally spaced portions. Setting the number of elements n_x and n_y in plane directions, each element length is $h_{xi}=L_1/n_x$ and the corresponding width is $h_{yi}=L_2/n_y$. However, it is well known that the solution of (1), (3) and (4) generally exhibits singular behaviour close to the edges and corners of S [17]. For this reason, a regular discretization of S may not be able to correctly approximate surface displacements and/or pressures at edges and corners. In order to obtain more accurate results, it is common to use power graded meshes [18], [19], [20], which are characterized by small surface portions close to edges and corners and large surface portions close to the origin. Such a discretization is obtained by means of a grading exponent $\beta\geq 1$, and assuming a generic dimensionless coordinate t, which may represent both x and y directions, on the interval (-1/2,1/2), characterized by the vertices of the subdivisions following the subsequent expression:

$$t_j=\begin{cases}\dfrac{1}{2}\left[\left(\dfrac{2j}{n}\right)^{\beta}-1\right] & \text{for} \quad 0\leq j\leq n/2\\ -t_{n-j} & \text{for} \quad n/2<j\leq n\end{cases}. \qquad (13)$$

where n is the number of subdivisions of the interval.

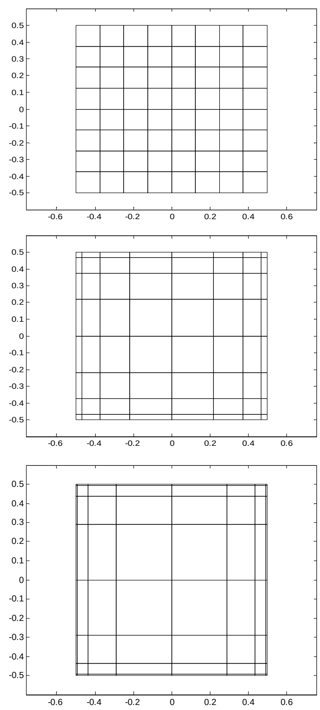


Fig.2 Examples of power graded surface discretization for a unitary square area, n=8 subdivisions along each side, varying β

For β=1, the vertices turn out to be equally spaced, but as β increases, the vertices are more concentrated at the end of the interval. In Fig.2 some examples of power-graded meshes are shown by considering for simplicity a square area having $L_1=L_2=1$ and with the same number of subdivisions along *x* and *y* directions. It is worth noting that for increasing β, the elements near the edges and corners of the discretized surface tend to be smaller and smaller, whereas, elements close to the origin tend to be bigger and bigger. Then, the exponent β in (13) has to be chosen in order to obtain accurate results both near surface edges and close to the origin.

4 Numerical tests

Several numerical tests are performed in order to evaluate the effectiveness and accuracy of the proposed numerical model. For instance, the problem of a rigid indenter on a 3D half-space is considered, together with the case of a uniform surface pressure. Results are compared with existing numeric and analytic solutions.

4.1 Rigid rectangular indenter on elastic half-space

A typical problem arising from (1) is related to the determination of the surface pressures generated by a well-defined displacement, namely a uniform rigid vertical displacement or a rigid rotation. Such a problem is also related to the determination of the translational and rotational stiffness of a rectangular foundation on the half-space. In the following sub-paragraphs, only square indenters are considered. Further details on rectangular indenters and convergence tests can be found in [16].

4.1.1 Square indenter subjected to a vertical displacement

An indenter subjected to a uniform vertical displacement is considered (Fig.3). This problem is frequently studied in soil-structure interaction analysis and it can be also defined as the uniform indentation of an elastic half-space by a smooth rigid rectangular footing.

Many researchers had already studied this problem by adopting different solution methods [6], [7], [8], [9], [10], [11], [12], [21]. The determination of the solution of the integral equation considered in (1) is also a classical problem in physics and represents the charge density of a thin, electrified square or rectangular plate *S* loaded by a given potential [18], [22]. This problem is also related to the determination of the dynamic translational stiffness of rigid foundations on elastic half-space [11], [23], [24]. Then, starting from (10) and considering different surface discretizations (varying β and increasing *n* along each side of the square foundation), the problem is numerically solved by adopting the proposed method. Results in terms of surface pressures are presented and the resulting translational stiffness is evaluated.

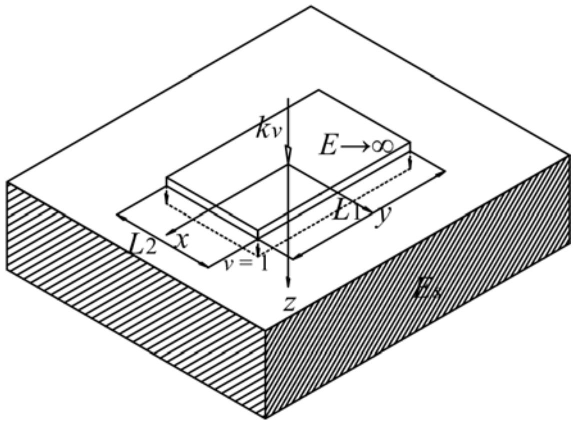


Fig.3 Rigid indenter on elastic half-space subjected to a uniform unitary vertical displacement

Setting a uniform unitary displacement value, the corresponding vector **q** is defined by putting each vector element equal to 1, then, from (10), the surface pressure vector is obtained:

$$\mathbf{r} = \mathbf{G}^{-1}\mathbf{H}\mathbf{q} \tag{14}$$

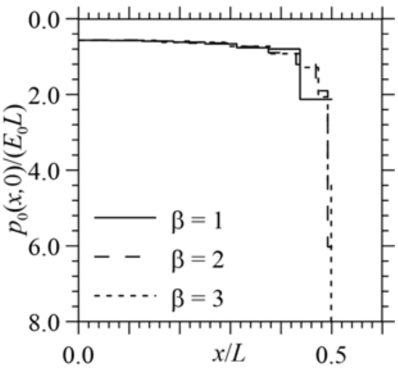
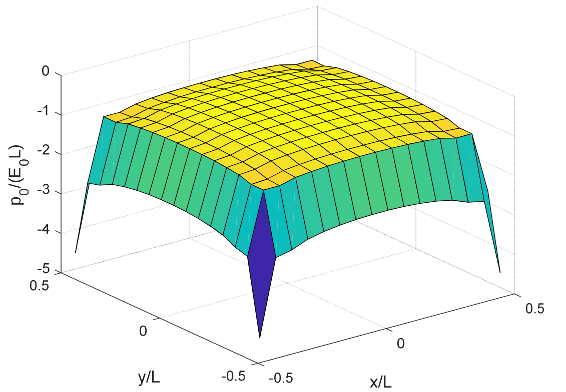
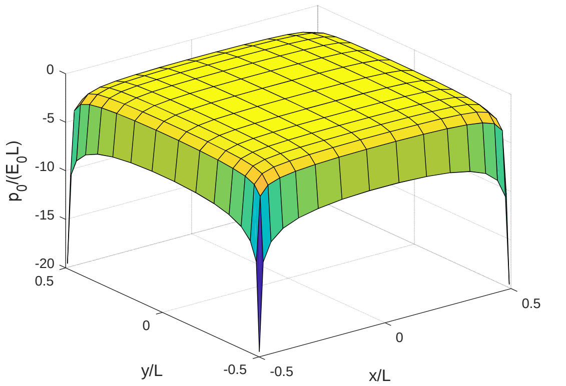


Fig.4 Dimensionless half-space surface pressures along *x* axis generated by a rigid square indenter with *n*=16 subdivisions along each side and varying β

Considering the case of a square indenter over a square surface ($L_1=L_2=L$), Fig.4 shows dimensionless surface pressures $p_0/(E_0L)$ along *x* axis, obtained with *n*=16 subdivisions and varying β. In Fig.5a,b,c, dimensionless surface pressures $p_0/(E_0L)$ are shown by adopting a three-dimensional representation. It can be noted that surface pressures assume an almost constant value quite close to the origin, whereas their absolute value increases rapidly in proximity of surface edges and corners.



a



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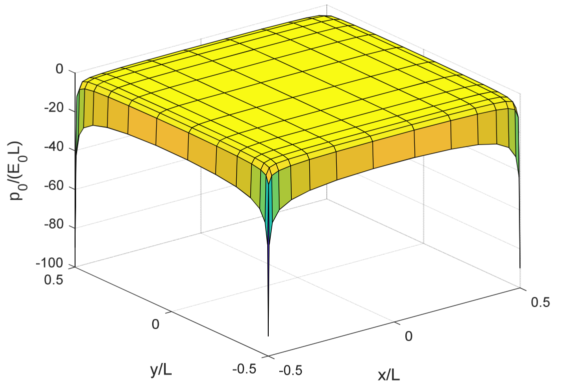

c

Fig.5 Dimensionless surface pressures generated by a unitary vertical displacement of a square surface subdivided with a power graded mesh having 16 subdivisions along each side and β=1 (a), 2 (b) and 3 (c)

As expected, results obtained with the uniform surface discretization are not able to represent correctly the behaviour at surface edges and corners, whereas increasing β, the pressure near surface edges and corners is able to better represent the singular behaviour typical of rigid stamps on elastic half-space.

The proposed numerical model is also able to evaluate the resultant of surface pressures **r** in order to determine the vertical translational stiffness k_v of a rigid square foundation on the half-space. Assuming an accurate surface discretization by adopting a power-graded mesh with β=4 and $n=2^7$, the proposed numerical model allows to obtain $k_v=1.1523E_0L$, which is in excellent agreement with existing numerical solutions (Table 1).

Table 1 Vertical translational stiffness value for a rigid square foundation.

Author	method	k_v/E_0L
present analysis	BEM	1.152
Guzina et al. [24]	BEM	1.152
Bosakov [12]	orthogonal polynomials	1.146
Erwin et al. [22]	BEM	1.152
Dempsey and Li [11]	numerical integration	1.152
Pais and Kausel [23]	-	1.175
Whitman and Richart [25]	-	1.080
Gorbunov and Posadov [7]	power series	1.095

4.1.2 Square indenter subjected to a rigid rotation

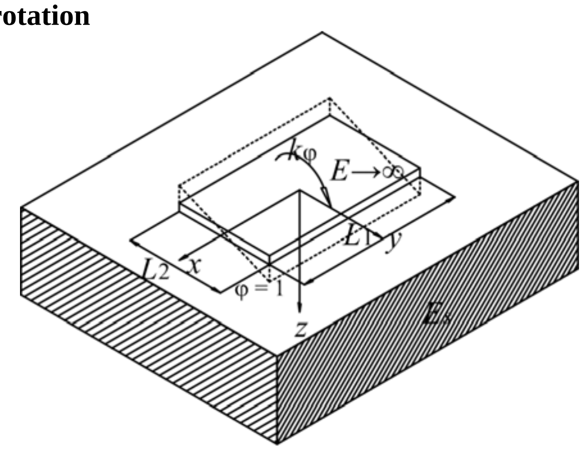

Fig.6 Rigid indenter on elastic half-space subjected to a unitary rotation with respect to *x* axis

Another problem arising from Eq.1, which is similar to the one described in the previous paragraph, consists in the determination of the rotational stiffness of a rigid indenter on the half-space with respect, for instance, to the *x* axis.
For this purpose, a unitary rotation value φ with respect to *x* axis is defined (Fig.6) and the corresponding displacement vector $\mathbf{q}_{\varphi x}$ is generated by multiplying the rigid rotation value for the distance of the centre of each surface portion with respect to *x* axis. The surface pressures caused by $\mathbf{q}_{\varphi x}$ are evaluated with (14) and the corresponding resultant bending moment with respect to *x* axis is

determined in order to obtain the rotational stiffness of the foundation.
Considering for simplicity a square foundation having $L_1=L_2=L$ and adopting the accurate surface discretization already used in the previous paragraph, the rotational stiffness turns out to be $k_{\varphi x}=0.2601E_0L^2$.

4.1.2 Rectangular indenters

The numerical results obtained in the previous paragraphs can be extended by considering rectangular indenters with varying L_1/L_2 ratio. In this case, the contact surface is discretized with a power graded mesh characterized by β=3 and $n_x=n_y=2^6$ is adopted. Results in terms of vertical translational stiffness and rotational stiffness with respect to *x* axis are shown in Fig.7 with crosses, by adopting the notation defined in the following expressions:

$$\beta_v = \frac{k_v}{E_0}\frac{1}{\sqrt{L_1L_2}}, \quad (15)$$

$$\beta_\varphi = \frac{k_\varphi}{E_0}\frac{1}{L_1L_2^2}, \quad (16)$$

in order to compare results with the data determined in [25] and [26]. Similar results adopting power series were obtained also in [7]. The present model turns out to be effective also for rectangular surfaces and the power graded mesh with β=3 turns out to be sufficient for obtaining accurate results.

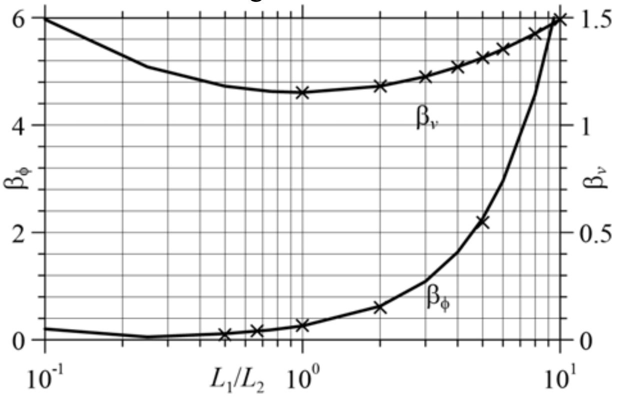

Fig.7 Vertical translational stiffness β_v and rotational stiffness β_φ of a rigid rectangular indenter on an elastic half space, varying L_1/L_2 ratio. Crosses for the present analysis, continuous lines for Whitman and Richart data [25]

4.2 Uniform pressure over a rectangular surface

Another problem arising from Eq. 1 consists in the determination of the vertical surface displacement generated by a uniform pressure *p* applied to a generic rectangular area of the half-space surface (Fig.8).

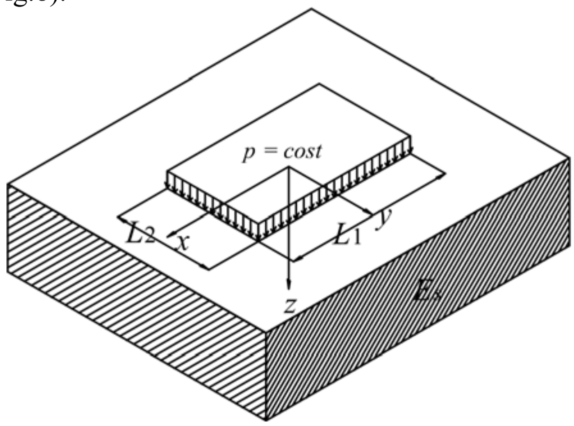

Fig.8 Elastic half-space loaded by a uniform vertical pressure over a rectangular area

The analytic solution of this problem was obtained by Love [5]. Here, the proposed numerical model allows to obtain the vector of surface vertical displacements generated by the vector of uniform surface pressures **r**=**p** as follows:

$$\mathbf{q} = \mathbf{H}^{-1}\mathbf{G}\mathbf{p} \quad (17)$$

Numerical results are compared with analytic results in terms of vertical surface displacements (Fig.9) evaluated at surface corner (C), center (O), and edges midpoint (M, N). By adopting a loaded surface discretization characterized by β=3 and $n_x=n_y=2^6$, numerical results for varying L_1/L_2 turn out to be in excellent agreement with analytic solutions.

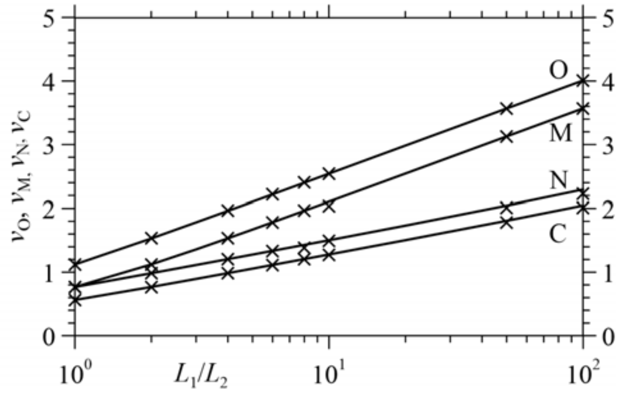

Fig.9 Vertical surface displacements under a rectangular area due to a uniform pressure, continuous lines for present analysis, crosses for Love solution [5]

5 Conclusion

In this work, a simple and effective numerical model has been proposed for applying the Galerkin Boundary Element Method to the solution of problems related to half-space surface subjected to vertical displacements and pressures. The numerical model is based on the discretization of both surface vertical displacements and pressures by means of a piecewise constant function. For this purpose, the half-space surface area subjected to pressures or displacements has been discretized successfully by means of a power graded mesh with rectangular subdivisions, instead of adopting a regular discretization, in order to better approximate the pressure values near corners and edges in case of rigid indenters on half-space. Several numerical tests have been performed in order to evaluate the effectiveness of the numerical model proposed. Results in terms of vertical translational stiffness of a square foundation turned out to be in excellent agreement with existing results, and, similarly, vertical displacements generated by a uniform pressure over a rectangular area turned out to be in excellent agreement with the existing analytic solution [5]. The effectiveness of the model, which has been already used for studying elastic beams on 3D half-space [15], will allow to numerically study other problems related to the interaction of shallow foundations with the elastic media.

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